Part I

Online Approaches
2

A DYNAMIC AMBULANCE MANAGEMENT MODEL FOR RURAL REGIONS

In this chapter, we consider the dynamic relocation problem, in which ambulances can be redeployed proactively throughout the region, for rural regions specifically. This type of region differs in a few aspects with respect to the urban case, e.g., the fleet size, the number of events and the spatial demand distribution. We construct a discrete-time Markov decision process (MDP) to model the Dynamic Ambulance Management (DAM) problem in an EMS system. Therefore, we model the road network of the region of interest as an equidistant graph and we take into account the current status of both the system and the ambulances in a state. We do not require ambulances to return to a base station: they are allowed to idle at any node in the graph. Instead, a policy is sought that specifies for each ambulance that is not busy with or en-route to a call whether to move to an adjacent node.

Since the MDP model is not tractable for large model instances, we present a heuristic approach to compute such redeployment actions. We construct several scenarios that may occur one time-step later and combine these scenarios with each feasible action to obtain a classification of actions. We show that on both patient- and crew-related performance indicators, the heuristic policy significantly outperforms a commonly used relocation policy structure in practice: the compliance table policy. Moreover, we compare the heuristic to the optimal policy for small-scaled instances of the problem, i.e., for an EMS system with few nodes and few ambulances.

The work in this chapter is based on Van Barneveld et al. (2015).

2.1 Introduction

The focus on rural regions has several important implications. In most papers on DAM, the numerical results section, in which the performance of the proposed
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methods is validated, is based on ambulance service providers operating in urban EMS regions, i.e., in large cities. However, there are substantial differences between urban and rural regions. We list three of them.

1. In rural regions, the number of ambulances is small compared to urban regions. Therefore, the effect of one ambulance fewer available, for instance, if this ambulance is busy, is more noticeable in rural regions with a limited number of ambulances. In contrast, in urban regions one ambulance fewer available probably only has a small impact on the ability to respond to emergency requests quickly. Thus, in rural regions, one has to be more careful about how to (re)deploy ambulances.

2. Besides, in rural EMS regions, the fluctuation in demand per area is much higher. There are areas with practically no demand, while in other areas, especially in cities or towns, the demand is high. As a consequence, an ambulance driving from a high-demand area to another high-demand area usually traverses an area of low demand, providing only very marginal coverage when it is en route. This is typically not the case in urban areas, in which an ambulance is always supplying coverage to a large amount of population, wherever it is. In this sense, relocating ambulances between areas of high demand involves more risks regarding the timely response to a patient.

3. A last difference between rural and urban regions is the number of events. Most papers, e.g., the work done by Gendreau et al. (2006), by Maxwell et al. (2010) and by Schmid (2012), assume that relocation decisions are taken only at the time of events, e.g., when the number of available ambulances changes due to a vehicle dispatch or service completion. This may work well for urban areas: after all, there are a lot of events due to the large number of demand requests. Hence, the moments at which dispatchers have the possibility to adjust the ambulance location plan, are numerous as well. In contrast, the number of events in rural regions is low, resulting in fewer opportunities to do this. Summarizing, urban and rural regions differ much from each other, and thus, they should be approached differently.

The structure of this chapter is as follows. In Section 2.2, we provide an MDP formulation for the ambulance relocation problem. In this problem, ambulances can be present at designated locations in the region: nodes in the graph representing the region of interest. The objective is to find a good ambulance configuration: a distribution of ambulances throughout the region in such a way that one is able to respond to an incoming request quickly. This ambulance configuration can be achieved by moving ambulances over the graph. We decide on how we should move these ambulances, given the state of the system. Moreover, a certain penalty is associated with each possible response time. This penalty is defined using penalty functions. The proposed MDP-formulation is not tractable for large problem instances, so we resort to a heuristic, which is the topic of Section 2.3. The general idea of the heuristic is to consider scenarios that may occur one time step later. We combine these scenarios with each possible change in ambulance configuration to obtain a potentially new state. In this state, we consider the minimal expected
penalty related to the response to additional requests and classify the movement, based on these expectations and the probability that this particular scenario occurs. We conclude this chapter by the numerical study in Section 2.4. This study is based on simulation results for an ambulance service provider in a rural EMS region in the Netherlands: Flevoland.

2.2 Model

In this chapter, we make some assumptions on the general EMS process as described in Section 1.1 in order to fit into our modeling framework. Each incoming request of a patient needs an ambulance to attend to. We assume that the level of priority of requests for an ambulance is equal for each request. That is, we only consider emergency requests of the highest urgency: the life-threatening A1-calls. This assumption is justified by the fact that ambulance service providers are mostly judged on their performance regarding the highest priority incidents. Upon arrival at the emergency scene, the ambulance crew decides whether the patient needs transportation to a hospital. We assume that this decision is made quickly after arrival at the emergency scene, since the crew is already informed of the severity of the request by the emergency control center agent. With probability $r$, $0 \leq r \leq 1$, a patient needs transportation. If so, the ambulance crew treats the patient for a random number of time units at the emergency scene: the treatment time on scene. Then, he/she is transported to the nearest hospital. There, the ambulance transfers the patient for some random time, which we will call treatment at hospital. We assume no queueing takes place at the hospital: emergency departments have infinite capacity. This assumption is justified by the fact that we focus on rural regions with a small number of incoming requests per hour. Summarizing, our simplified EMS process is as follows: when the ambulance arrives at the emergency scene, the remaining time the ambulance is busy consists of a stochastic treatment time on scene, a deterministic transportation time, and a stochastic treatment time at the hospital. We refer to these stages as phase 1 to phase 4, see Figure 2.1.

Note that we do not include a phase for ambulances that are on their way to respond to a patient. The reason for this is that such an ambulance, although initially assigned, may not be the one to provide service. This kind of behaviour
occurs if a second ambulance, located closer to the request, becomes idle when the first one is en route. Therefore, as long as an ambulance is on its way to an emergency request, it is regarded as if it is idle. If a patient does not need transportation to a hospital, the busy time of the ambulance only consists of the stochastic treatment time on scene.

We model the region of interest as a graph, with $\mathcal{N}$ as its node set. The nodes of the graph serve as demand locations: locations where an incident might occur. There are two types of nodes: nodes with and without a hospital. Let these disjoint sets be denoted by $\mathcal{H}$ and $\mathcal{H}$, respectively, where $\mathcal{N} = \mathcal{H} \cup \mathcal{H}$. For simplicity, we enumerate the nodes in such a way that there is a hospital at the first $|\mathcal{H}|$ nodes in the enumeration, so

$$\mathcal{N} = \{1, 2, \ldots, |\mathcal{H}|, |\mathcal{H}| + 1, \ldots, |\mathcal{H}| + |\mathcal{H}|\}.$$  

The road network is modelled by edges, that can be either one- or bidirectional, depending on whether a U-turn is allowed on the specific road. This is typically not the case on highways. We assume that the length of each edge equals 1, so it takes one time step to traverse an edge. Therefore, time is discretized in time steps of $\Delta t$. As a consequence, it takes an ambulance $\Delta t$ time units (e.g., 5 minutes) to cross an edge. In realistic situations, the graph is constructed in a way that $\Delta t$ is fine enough to model ambulance movements. To model more realistic situations, one could decrease $\Delta t$, but then the graph should contain more nodes and edges. Therefore, for $\Delta t \to 0$, this model becomes continuous in both time and space.

Moreover, another assumption made in this model is that the number of in-
coming calls at each demand location per unit of time is Poisson distributed with parameter $p_i(\Delta t)$ for node $i$, $i \in \mathcal{N}$. These parameters can easily be estimated using historical data. In reality, these parameters vary over time, but here we assume that these are fixed for the sake of simplicity. Moreover, this is not really a limitation, since one can use different parameter values for different times of the day. For modeling issues, we assume that no external requests arrive at hospitals, so $p_h(\Delta t) = 0$ for $h \in \mathcal{H}$. The total number of ambulances in the system is $A$ and all ambulances are of same type. The fleet size is homogeneous, constant and does not vary over time. Although $p(\Delta t) = (p_i(\Delta t))_{i \in \mathcal{N}}$ depends on the chosen time step size $\Delta t$, we will omit this dependence in the remainder for readability issues. The notation is summarized in Table 2.1.

### 2.2.1 State Space

There are four major sources of randomness in the EMS process model considered in this chapter: the arrival of requests, the possible need for transportation to a hospital, the service time on scene, and the time an ambulance spends at the hospital, see Figure 2.2. The state of our system in our MDP formulation is given by five components, which we describe in detail below. For an overview of the notation of the state space variables, we refer to Table 2.2.

#### Table 2.2: State space variables.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$</td>
<td>Number of patients at location $i$.</td>
</tr>
<tr>
<td>$y_i$</td>
<td>Number of ambulances at location $i$.</td>
</tr>
<tr>
<td>$b_i$</td>
<td>Number of busy ambulances at location $i$.</td>
</tr>
<tr>
<td>$f_i$</td>
<td>Number of idle ambulances at location $i$.</td>
</tr>
<tr>
<td>$z_i$</td>
<td>Number of ambulances at location $i$ that need to transport a patient.</td>
</tr>
<tr>
<td>$Z(k, j)$</td>
<td>Elapsed treatment time of ambulance $j$ at node $k$, $k \in \mathcal{N}$.</td>
</tr>
<tr>
<td>$D(h, t)$</td>
<td>Number of occupied ambulances that will arrive at hospital $h$ in $t$ time units.</td>
</tr>
</tbody>
</table>
1. **The number of patients per demand location.** Due to the spatial aggregation, there can be multiple patients waiting for an ambulance at the same time in the same area. This number is denoted by a vector \( x = (x_1, x_2, \ldots, x_N) \) of length \( N = |\mathcal{N}| \), where \( x_i \in \mathbb{N}_0 \) for \( 1 \leq i \leq N \). We assume that each patient needs an ambulance and an ambulance cannot serve more than one patient at a time.

2. **The number of ambulances either in phase 1, 2 or 4 per demand node.** This is similarly denoted as the previous state component by \( y = (y_1, y_2, \ldots, y_N) \) of \( N \), where \( y_i \in \mathbb{N}_0 \), \( 1 \leq i \leq N \). Moreover, \( \sum_{i=1}^{N} y_i \leq A \), since the fleet size cannot be exceeded. If there is both a patient and an ambulance at a certain node, we assume that this ambulance is treating this patient: the vector \( b = (b_1, b_2, \ldots, b_N) \) of busy ambulances (ambulances either in phase 2 or phase 4) is given by \( b = \min(x, y) \). In addition, \( f = y - b \) denotes the vector of idle ambulances, i.e., the ambulances in phase 1.

3. **The number of ambulances per demand location required to transport patients.** That is, the number of phase 2 ambulances that once the treatment on scene has finished, will make a transition to phase 3. We denote this by a vector \( z = (z_1, z_2, \ldots, z_N) \), where \( 0 \leq z_i \leq b_i \) for each node \( i \). Moreover, an ambulance at a hospital does not have to transport a patient, so \( z_h = 0 \) for \( h \in \mathcal{H} \).

4. **The elapsed service time of ambulances in phases 2 and 4.** We denote this by a matrix \( Z \) with \( |\mathcal{N}| \) rows and \( A \) columns, where \( Z(k, j) \) denotes the elapsed service time of ambulance \( j_1 \) at node \( k \), where \( j_1 \leq b_k \). Moreover, \( Z(k, j_2) = -1 \) for \( b_k < j_2 \leq A \). Hence, \( \sum_{j=1}^{A} \mathbb{1}(Z(k, j) \geq 0) = b_k \). We assume that each row of \( Z \) is sorted in non-increasing order, in order to simplify the description of the computations in Sections 2.2.3 and 2.3.1. Rows \( k \leq |\mathcal{H}| \) and the remaining rows correspond to ambulances treating at hospitals and ambulances treating on scene, respectively.

5. **Destinations and remaining driving times of ambulances in phase 3.** We denote these quantities by a matrix \( D \). Let \( D(h, t) \) describe the number of phase-3 ambulances that will arrive in \( t \geq 1 \) time units at hospital \( h \in \mathcal{H} \). Note that \( \sum_{h \in \mathcal{H}} \sum_{t=1}^{L} D(h, t) + \sum_{i=1}^{N} y_i = A \), where \( L \) denotes the length of the longest path that any ambulance might take.

A state \( s \) is now defined by the tuple \( s = (x, (b, f), z, Z, D) \), or equivalently: \( s = (x, y, z, Z, D) \), where \( y = b + f \).

### 2.2.2 Actions

We will now describe the control process of the MDP. An action set belongs to each state \( s \). Actions describe the change in configuration of idle ambulances: we can either dispatch an idle ambulance to one of its neighbouring nodes, or we can
let it hold its current position. An action belonging to the action set of state $s$ is denoted by

$$a^s = (a_1^s, a_2^s, \ldots, a_N^s),$$

where $a_i^s \in \mathbb{Z}$ denotes the change in $y_{i,s}$, i.e., the change in the number of ambulances present at node $i$. It is possible that $a_i^s = 0$, while ambulances are moving from/to node $i$. This occurs when the number of incoming ambulances equals the number of outgoing ambulances at location $i$. To keep track of the exact movement of ambulances, we can decompose $a^s$ into an $(a_i^s)^-$ and an $(a_i^s)^+$-part, where $(a_i^s)^-$ and $(a_i^s)^+$ denote the number of outgoing and incoming ambulances per node, respectively. Naturally, $(a_i^s)^-, (a_i^s)^+ \in \mathbb{N}_0$ for $i \in \mathcal{N}$ and $a^s = (a_i^s)^+ - (a_i^s)^-$. Action $a^s$ satisfies the condition $-a_i^s \leq f_i$, since no more than $f_i$ ambulances can be removed from node $i$. Similarly, no more than the total number of idle ambulances can be sent to location $i$, so $(a_i^s)^+ \leq \sum_{j \neq i} f_j$. All edges have length 1, so it takes exactly one time step to carry out an action. Therefore, it holds that

$$\sum_{i=1}^{N} ((a_i^s)^+ - (a_i^s)^-) = \sum_{i=1}^{N} a_i^s = 0,$$

since the number of departing idle ambulances equals the number of arriving idle ambulances. Furthermore, since the actions are configuration-based rather than based on each ambulance separately, idle ambulances are indistinguishable.

Note that actions are only defined for idle ambulances. Busy ambulances, which are ambulances either treating a patient at an emergency scene or at a hospital, continue their service. There are actions that are not reasonable to take, but still allowed: actions in which the response time to a request is unnecessarily delayed. We want to exclude these actions since these are suboptimal in the model and in reality they are not even considered. We call these actions infeasible. The question arises on how to define the set of feasible actions, which we denote by $\mathcal{F}(s)$ for state $s$. To compute $\mathcal{F}(s)$ in state $s$, we solve either a minimum weighted bipartite matching (MWBM) problem or a linear bottleneck assignment problem (LBAP). Both problems differ in objective, but they share the same modeling framework, which we describe next in the context of the ambulance relocation model considered in this chapter.

Assume $s = (x, (b, f), z, Z, D)$. Let $F = \sum_{i=1}^{N} f_i$ and $X = \sum_{i=1}^{N} (x_i - y_i)^+$ denote the number of idle ambulances and the number of requests that are not served by an ambulance yet, respectively. We introduce a weighted complete bipartite graph $K_{F,X} = (V_1 \cup V_2, E, l)$, where $V_1, V_2$ are the two node sets, $E$ the edge set and $l$ a function assigning weights to edges. The node set $V_1$ corresponds to the locations of the $F$ idle ambulances: for each ambulance we introduce a node indexed by its location. In a similar way, we define the node set $V_2$, but these nodes correspond to the location of patients waiting. If there are more ambulances or patients waiting at a particular location, then we specify the nodes belonging to this location with a subindex. Let $v_1 \in V_1, v_2 \in V_2$. The weight $l((v_1, v_2))$ of edge $(v_1, v_2)$ equals the length of the shortest path between $v_1$ and $v_2$. This corresponds to the required number of time units to travel from the corresponding locations of $v_1$ and $v_2$ in the original graph representing the region of interest. Therefore,
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Moreover, we define a matching as a set of edges without common nodes. A node is matched if it is an endpoint of one of the edges in the matching. A matching is called maximal if all nodes in \( V_1 \) or all nodes in \( V_2 \) are matched. Both the MWBM and the LBAP aim to find an optimal matching. We explain both assignment problems next.

**Minimum Weighted Bipartite Matching**

It seems obvious to always dispatch the nearest ambulance to a request. However, this action can be suboptimal in our model, because it possibly delays the response time to a different request. By modeling the assignment problem by an MWBM, the total response time to all requests that are not served yet is minimized. If there is only one such request, this assumption is equivalent to the policy in which the nearest ambulance is assigned to a request. A minimum weighted bipartite matching is defined as a maximal matching \( M \) where the sum of the weights of the edges in \( M \) has a minimal value. That is, our objective criterion is

\[
\min_{M \in \mathcal{M}} l(M) = \min_{M \in \mathcal{M}} \sum_{e \in M} l(e) \quad (2.1)
\]

and \( \mathcal{M} \) is the set of all maximal matchings. The assignment problem is solved by the Hungarian Algorithm, which runs in \( \mathcal{O}((|V_1| + |V_2|)^2|E|) \) time (Schrijver, 2003). Note that finding a minimum weighted bipartite matching in \( K_{F,X} \) is equivalent to finding an assignment of ambulances to requests with respect to Equation (2.1).

**Linear Bottleneck Assignment**

The objective of minimizing the total mean response time to all patients that are waiting seems reasonable. However, one can argue about it. Possibly, an ambulance responds to the majority of these patients within a short amount of time, while for one patient it takes a very long time before an ambulance arrives at the emergency scene. It seems fairer to divide the total response time equally over all requests waiting. A way to achieve this is to model the matching problem as a linear bottleneck assignment problem (LBAP). Instead of minimizing the total sum of the edges in the matching as before, the LBAP aims to find a maximal matching with the property that the maximum weight of the edges in the matching is minimized. That is, the objective criterion is

\[
\min_{M \in \mathcal{M}} l(M) = \min_{M \in \mathcal{M}} \max_{e \in M} l(e),
\]

and \( \mathcal{M} \) is the set of all maximal matchings. This problem and several of its solution methods are treated in detail in Burkhard et al. (2009), in which a polynomial-time algorithm is proposed. Moreover, if the set of such matchings contains more than one such matching, this algorithm finds the matching with minimal total weight in this set. In the context of dynamic ambulance management, this translates to obtaining an assignment of idle ambulances to requests, such that the maximum response time is minimized and given this maximum response time, the total response time is minimized.
### Table 2.3: Types of randomness in the evolution of the system.

<table>
<thead>
<tr>
<th>Random Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1^i$</td>
<td>Number of arriving requests at location $i$.</td>
</tr>
<tr>
<td>$\omega_2^i(s)$</td>
<td>Number of treatment completions on scene in state $s$ at location $i$.</td>
</tr>
<tr>
<td>$\omega_3^h(s)$</td>
<td>Number of treatment completions at hospital $h$ in state $s$.</td>
</tr>
<tr>
<td>$\omega_4^i(s, \omega_2^i(s))$</td>
<td>Number of ambulances departing for a hospital from node $i$ in state $s$.</td>
</tr>
<tr>
<td>$\omega_5^i(s, \omega_1^i, \omega_4^i(s, \omega_2^i(s)))$</td>
<td>Number of patients at location $i$ decided to be transported.</td>
</tr>
</tbody>
</table>

Now, the construction of the set of feasible actions in state $s$, $\mathcal{F}(s)$, is as follows: we solve either an MWBM or an LBAP to obtain a matching in which ambulances are assigned to requests. If the number of patients waiting is smaller than the number of idle ambulances, there are ambulances that are not assigned to requests. For these ambulances, we have a choice where to send them to. For these states, the set of feasible actions contains more than one action: if $X < F$ in state $s$, we have to decide for $F - X$ ambulances how to relocate them, where ambulances that are not assigned to a request can either be relocated to one of the neighbouring nodes or they can keep their position. However, if the number of patients waiting exceeds the number of idle ambulances, we can not respond to all requests. Then, we have only one possible action: the action induced by the matching. This is also the case when we have an equal number of idle ambulances and patients waiting.

#### 2.2.3 Evolution

In this section, we describe the underlying dynamics of the MDP model of the EMS system considered in this chapter. If our current state is $s = (x, y, z, Z, D)$ and action $a^s \in \mathcal{F}(s)$ is taken, the system evolves according to random variables related to number of arriving requests (denoted by $\omega_1$), number of treatment completions ($\omega_2$ and $\omega_3$), number of ambulances departing to a hospital ($\omega_4$) and the number of patients for which it is decided that they need transportation ($\omega_5$). These random variables are summarized in Table 2.3. We describe the dynamics per state component. Let $s' = (x', y', z', Z', D')$ denote the next state.

1. **The number of patients per demand location.** We distinguish between nodes with and nodes without a hospital. Consider node $h \in \mathcal{H}$. The number of requests at hospital $h$ in the next state, $x'_h$, depends on two processes: arrival of occupied ambulances at node $h$ and the completion of treatments by an ambulance at hospital $h$. Recall that $p_h = 0$ for $h \in \mathcal{H}$, so there are no new arrivals. The number of arriving ambulances at location $h \in \mathcal{H}$ in the next time step equals $D(h, 1)$. For the number of completions, we use the Poisson binomial distribution, which is the discrete probability distribution of a sum of independent Bernoulli
trials that are not identically distributed (Wang, 1993). The probability of having \( \kappa \) successful trials out of a total of \( n \) trials can be written as the sum
\[
\mathbb{P}\{K = \kappa\} = \sum_{U \in \mathcal{U}_n} \prod_{i \in U} \rho_i \prod_{j \in \bar{U}} (1 - \rho_j),
\]
(2.2)
where \( \rho^1, \rho^2, \ldots, \rho^n \) denote the success probabilities, \( \mathcal{U}_n \) is the set of all subsets of \( \kappa \) integers selected from \( \{1, 2, \ldots, n\} \). The ordinary binomial distribution is a special case where all the success probabilities are equal. The number of completions, denoted by \( \omega^3_i(s) \), is a Poisson binomially distributed number on \( b_h \) trials. The success probability \( \rho^j_h \), which is the probability that ambulance \( j \) at \( h \) will have finished its treatment at the next step, depends on the elapsed service time of ambulance \( j \), which is \( Z(h, j) \). That is,
\[
\rho^j_h = \mathbb{P}\{B^j_h = Z(h, j) + 1|B^j_h > Z(h, j)\},
\]
(2.3)
where \( B^j_h \) is the treatment time of ambulance \( j \) at hospital \( h \). The \( b_h \) probabilities needed for the Poisson binomial distribution are given by the vector \((\rho^1_h, \rho^2_h, \ldots, \rho^{b_h}_h)\). Now,
\[
x'_h = x_h + D(h, 1) - \omega^3_i(s), \quad h \in \mathcal{H}.
\]

If \( i \in \mathcal{H} \), i.e., if there is no hospital at node \( i \), then \( x'_i \) is defined differently, since it depends on two other processes: the arriving requests at location \( i \) and the number of treatment completions on scene at location \( i \). These numbers are denoted by \( \omega^1_i \) and \( \omega^2_i(s) \). Note that \( \omega^1_i \) does not depend on \( s \) and is Poisson distributed with parameter \( p_i \). In contrast, \( \omega^2_i(s) \) does depend on \( s \): this number is Poisson binomially distributed with success probability \((\rho^1_i, \rho^2_i, \ldots, \rho^{b_i}_i)\).

2. The number of ambulances either in phase 1, 2 or 4 per demand location. For the evolution of \( y = f + b \), we also distinguish between hospital locations and other locations. We consider the case that \( i \in \mathcal{H} \) first. The number of ambulances \( y'_i \) in the next state at location \( i \) depends on the current number of ambulances \( y_i \), the action \( a^i_s \) and the number of ambulances that completes service on scene and departs for a hospital. Let this random number, which depends on the number of completions on scene, be denoted by \( \omega^4_i(s, \omega^2_i(s)) \). Then, we find that
\[
y'_i = y_i + a^i_s - \omega^4_i(s, \omega^2_i(s)), \quad i \in \mathcal{H}.
\]
The quantity \( \omega^4_i(s, \omega^2_i(s)) \) is determined as follows. We have \( b_i \) ambulances that are serving a patient on scene. Of these \( b_i \) ambulances, \( z_i \) ambulances need to go to a hospital and \( \omega^2_i(s) \) ambulances complete their service on scene now. Therefore, \( \omega^4_i(s, \omega^2_i(s)) \) is hypergeometrically distributed on a population size of \( b_i \) of which \( z_i \) are of one type, and \( b_i - z_i \) of the other type. Moreover, the number of draws is \( \omega^2_i(s) \).

If \( h \in \mathcal{H} \), \( y'_h \) depends on the current number of ambulances at location \( h \), the action \( a^*_h \), and the number of occupied ambulances that arrive at hospital \( h \). Hence,
\[
y'_h = y_h + a^*_h + D(h, 1), \quad h \in \mathcal{H}.
\]
3. The number of busy ambulances per demand location required to transport patients. When considering the number of busy ambulances at hospitals required to transport patients, it is clear that \( z_h = 0 \) for \( h \in H \). If \( i \) does not correspond to a hospital location, \( z_i \) is obtained as follows. It depends on \( \omega^4_i(s, \omega^2_i(s)) \) defined before and the number of new patients for which it is decided that they need transportation. Let this last random quantity be denoted by \( \omega^5_i(s, \omega^1_i(s, \omega^2_i(s))) \). Note that this number depends on the number of arriving and completed requests \( \omega^1_i(s, \omega^2_i(s)) \). Then,

\[
z'_i = z_i - \omega^4_i(s, \omega^2_i(s)) + \omega^5_i(s, \omega^1_i(s, \omega^2_i(s)))
\]

Note that this number is bounded by the number of ambulances that start a treatment of a new patient at location \( i \), denoted by \( d_i \). Then,

\[
d_i = \min\{\omega^2_i(s) - \omega^4_i(s, \omega^2_i(s)) + f_i + a^s_i, \omega^1_i + x_i - b_i\}, \tag{2.4}
\]

where \( \omega^2_i(s) - \omega^4_i(s, \omega^2_i(s)) + f_i + a^s_i \) equals the number of ambulances that start a new treatment: there were already \( f_i \) idle ambulances and we add the \( \omega^2_i(s) \) ambulances that complete service. However, \( \omega^5_i(s, \omega^2_i(s)) \) of these ambulances leave for a hospital and cannot start a new treatment. If \( a^s_i > 0 \), we have arrivals of ambulances, which can all start a new service, so we add that number as well. If \( a^s_i < 0 \), some of these idle ambulances leave for a different location and these ones cannot start a treatment at location \( i \). Note that

\[
\omega^2_i(s) - \omega^4_i(s, \omega^2_i(s)) + f_i + a^s_i \geq 0,
\]

since \( \omega^2_i(s) - \omega^4_i(s, \omega^2_i(s)) \geq 0 \) and \( f_i + a^s_i \geq 0 \). However, not all of these \( \omega^2_i(s) - \omega^4_i(s, \omega^2_i(s)) + f_i + a^s_i \) ambulances can start a new service if there are not that many requests waiting at \( i \). This quantity is given by \( \omega^1_i + x_i - b_i \): there were \( x_i - b_i \geq 0 \) patients without an ambulance treating them, and \( \omega^1_i \) additional requests arrive. Then, \( \omega^5_i(s, \omega^1_i(s, \omega^2_i(s))) \) is binomially distributed on \( d_i \) ambulances that start a new treatment, and with probability \( r \) each of these patients requires transport.

4. The elapsed service time of ambulances in phases 2 and 4. Let \( h \in H \) and let \( Z(h) \) denote the \( h \)-th row of \( Z \). The evolution of \( Z(h) \) depends on two processes: the completion of the service time of patients and the arrival of occupied ambulances. The number of completions at hospital \( h \) is \( \omega^3_h(s) \). Each busy ambulance \( j \) completes its service with probability \( \rho^j_h \) defined in (2.3), where \( j \leq b_h \).

Thus, in total there are \( I = \binom{b_h}{\omega^3_h(s)} \) options, which we enumerate by the variable \( i \), for the new configuration of busy ambulances at \( h \). Each of these options has positive probability. To calculate these probabilities, we need to enumerate all options. Define \( U_{b_h}(\omega^3_h(s)) \) as the set of subsets of \( \omega^3_h(s) \) integers that can be selected from \( \{1, 2, \ldots, b_h\} \). Moreover, let \( U^i \in U_{b_h}(\omega^3_h(s)) \) be the set of ambulances that remain busy in the \( i \)-th option, where \(|U^i| = b_h - \omega^3_h(s)\) and \( 1 \leq i \leq I \). Then we define \( \pi(U^i) \) as the probability that only the ambulances in \( U^i \) remain busy. These probabilities are calculated by

\[
\pi(U^i) = \prod_{j_1 \in U^i} (1 - \rho^j_h) \prod_{j_2 \in U^i} \rho^j_h,
\]
where $\bar{U}^i = \{1, 2, \ldots, b_h\} \setminus U^i$. This equals the probability mass function of the Poisson binomial distribution given in (2.2), but here we condition on $\omega^3_h(s)$. Therefore, $\sum_{i=1}^{I} \pi(U^i) < 1$, so we need to normalize. Let $\pi'(U^i)$ denote the normalized probabilities. That is,

$$
\pi'(U^i) = \frac{\pi(U^i)}{\sum_{i=1}^{I} \pi(U^i)}
$$

for each outcome $i$, $1 \leq i \leq I$. Now, we obtain a probability distribution on the set of outcomes and we sample an option from this distribution. Assume the sampled outcome is $i$. Then we define $Z^*(h)$ as follows:

$$
Z^*(h, j) = \begin{cases} 
-1 & \text{if } j \in \bar{U}^i \text{ or } b_h < j, \\
Z(h, j) + 1 & \text{if } j \in U^i.
\end{cases}
$$

(2.5)

If $j \leq b_h$ and $j \in \bar{U}^i$, the $j$-th ambulance completes its service and is no longer busy; its elapsed service time is discarded. In the second case in (2.5), the $j$-th ambulance does not finish its treatment and thus its elapsed service time is increased by 1 time unit. Then, we sort $Z^*(h)$ in non-increasing order to make sure that there are no $-1$’s in the first $b_h - \omega^3_h(s)$ entries.

Up to now, we only considered the completions of busy ambulances. However, occupied ambulances can arrive at hospital $h$ as well. Note that during the transition from $s$ to $s'$, $D(h, 1)$ ambulances arrive at $h$. Then, $b'_h = b_h - \omega^3_h(s) + D(h, 1)$ and

$$
Z'(h, j) = \begin{cases} 
Z^*(h, j) & \text{if } j \leq b_h - \omega^3_h(s), \\
0 & \text{if } b_h - \omega^3_h(s) < j \leq b'_h, \\
-1 & b'_h < j.
\end{cases}
$$

Recall that we conditioned on $\omega^3_h(s)$. Alternatively, we could have chosen to consider all $2^{b_h}$ options. That is, we do not condition on $\omega^3_h(s)$. If we define a probability distribution on all these $2^{b_h}$ options, the probabilities sum up to 1. Sampling from this distribution, we immediately obtain a new configuration and the number of completions defined as $\omega^3_h(s)$. For $i \in \mathcal{H}$, the evolution is similar, with $\omega^2_i(s)$ instead of $\omega^3_i(s)$ and no $D(h, 1)$-term.

5. Destinations and remaining driving times of ambulances in phase 3.

Let $H(h)$ describe the set of demand locations for which hospital $h$ is nearest among all hospitals. Formally,

$$
H(h) = \{i \in \mathcal{N} \mid l(i, h) \leq l(i, h') \forall h' \in \mathcal{H}, \ h \neq h\},
$$

where $l(i, h)$ denotes the required number of time units to travel from $i$ to $h$. Remember that we assume that a patient, who needs transportation, is always transported to the nearest hospital. However, it is possible that there exist two hospitals $h_1$ and $h_2$ for which $H(h_1) \cap H(h_2) \neq \emptyset$, i.e., there is a demand location for which these two hospitals are both closest. We aim to send all occupied ambulances
from this location to only one hospital, so we create a partition of the node set by using the recursion

\[ H^*(h) = H(h) \setminus \bigcup_{j=1}^{h-1} H^*(j). \]

That is, if multiple hospitals are nearest, we send all occupied ambulances to the first hospital according to the enumeration of the nodes. Then, \( D(h,t) \) evolves as follows:

\[ D'(h,t) = D(h,t+1) + \sum_{i \in \mathcal{H}} \omega^4_i(s, \omega^2(s)) \mathbb{1}_{\{i \in H^*(h)\}} \mathbb{1}_{\{l(i,h) = t\}}, \]

where \( \omega^4_i(s, \omega^2(s)) \) denotes the number of occupied ambulances departing for a hospital from location \( i \). The term \( \mathbb{1}_{\{i \in H^*(h)\}} \mathbb{1}_{\{l(i,h) = t\}} \) equals 1 if and only if \( h \) is the nearest hospital to location \( i \) and the travel time from \( i \) to \( h \) is \( t \).

### 2.2.4 Objectives

In practice, each country, possibly even each ambulance service provider within a country, uses its own performance measure. In this section we demonstrate how to incorporate different objectives in our MDP formulation. We do this by introducing a non-negative continuous penalty (or cost) function \( \Phi \), which is a function of the response time solely, with domain \( \mathbb{R}_{\geq 0} \). Several examples of cost functions are displayed in Figure 2.3.

Denote the cost in state \( s = (x,y,z,Z,D) \) by \( c(s) \). Let \( F = \sum_{i=1}^{N} f_i \) and \( X = \sum_{i=1}^{N} (x_i - y_i)^+ \) denote the number of idle ambulances and the number of requests that are not served by an ambulance yet, respectively. We solve an assignment problem as described in Section 2.2.2 to obtain an assignment of idle ambulances to the \( X \) waiting patients. Unless \( F < X \), each waiting request is assigned and a certain (remaining) response time to each of these requests is obtained. Denote these (remaining) response times by \( R^s_1, R^s_2, \ldots, R^s_X \) for an enumeration of waiting requests. Note that \( R^s_i > 0, i = 1, \ldots, X \). Now we define

\[ c(s) = \sum_{i=1}^{X} (\Phi(R^s_i) - \Phi(R^s_i - 1)). \] (2.6)

Note that the total penalty generated by request \( i \) equals

\[ \sum_{t=1}^{R^s_i} (\Phi(t) - \Phi(t-1)) = \Phi(R^s_i). \]

This is the case if the ambulance assigned to it is not reassigned to a different request, where \( R^s_i \) denotes the total response time to request \( i \). If the ambulance is reassigned, the penalty is slightly different. However, this hardly occurs in practice. If \( F < X \), that is, there are not enough idle ambulances to respond to each of the waiting patients, we set the response time to an unassigned request
equal to a large number. After some time steps this request will get assigned, but before that it generates costs as well. The objective is to minimize average costs over an infinite horizon.

An obvious performance measure is the average response time to a request. The objective of minimizing the average response time corresponds to a linear cost function:

$$\Phi(t) = t, \quad t \geq 0,$$

which is displayed in Figure 2.3a. Each additional time unit of delay generates the same penalty, since the derivative of this function is constant. Using this cost function results in a small average response time, but the variance may be large. Another commonly used type of performance measure is the percentage of emergency requests responded to within a certain maximum allowed response time threshold $T_{max}$, given by

$$\Phi(t) = \begin{cases} 0 & t \leq T_{max}, \\ 1 & t > T_{max}. \end{cases}$$

The penalty function corresponding to this performance measure is displayed in Figure 2.3b, and using it will relocate the ambulances in such a way that the coverage of the EMS region is maximized. However, in using this penalty function, there is no difference in penalty between a really short response time and a response
time that is slightly below the maximum allowed one. To overcome this problem, one could use the penalty function

\[ \Phi(t) = \frac{1}{1 + e^{-\beta(t-T_{\text{max}}-0.5)}}, \quad t \geq 0, \quad (2.9) \]

where \( \beta \geq 0 \) is a scaling parameter. This function is displayed in Figure 2.3c. This function is a smooth version of the function of Equation (2.8). The penalty function in Figure 2.3d has the interpretation of minimizing average lateness, and is given by

\[ \Phi(t) = \begin{cases} 
0 & t \leq T_{\text{max}}, \\
t - T_{\text{max}} & t > T_{\text{max}}.
\end{cases} \quad (2.10) \]

The function in Figure 2.3e, which is suggested by practitioners in the field, combines the cost functions in Figures 2.3a–2.3d and is given by

\[ \Phi(t) = \begin{cases} 
\frac{1}{\gamma}(e^t - 1) & 0 \leq t \leq T_{\text{max}}, \\
t - (T_{\text{max}} - 1) & t > T_{\text{max}},
\end{cases} \quad (2.11) \]

where \( \gamma \geq 0 \) is a scaling parameter. At \( T_{\text{max}} \), the function makes a jump to ensure that not meeting the maximum allowed response time is much worse than a response time that does. For \( t > T_{\text{max}} \), we use the performance measure of minimizing average lateness. To differentiate between response times before \( T_{\text{max}} \), we use an exponential cost function. Moreover, other penalty functions, for instance, penalty functions related to survival of a patient as considered in Erkut et al. (2008) and Chapter 5, can be incorporated.

### 2.2.5 Tractability

The state space is high-dimensional; in theory, we have infinitely many states, since there is no upper bound on the number of requests per location. However, we can introduce such an upper bound to obtain a finite number of states, but even for small-size instances solving the problem, i.e., finding the optimal policy, becomes intractable. This is not only a consequence of the high-dimensional state space: the large number of actions plays a role as well. This number can be very large for states with few requests, since we allow ambulances to move to each neighbouring node, not only to designated nodes such as base stations. As a consequence, solving this problem by modeling it as an MDP and applying methods described by Puterman (1994) is not tractable for realistic settings, although we were able to compute the optimal policy for a simplified example, c.f., Section 2.4.2. Therefore, we resort to a heuristic solution, which is the topic of Section 2.3.

### 2.3 Heuristic Solution

In this section we propose a heuristic that computes an action, given the state of the system, in order to overcome the tractability issues mentioned above. The general idea of this heuristic is to take the feasible action that minimizes the
expected penalty generated by an arriving request during the next time step, given the current state of the system. It is a one step look-ahead method that generates several scenarios that may occur one step later. All of these scenarios are possible outcomes of the evolution of the system, described in Section 2.2, with the action in which each idle ambulance keeps its position. However, we only generate scenarios in which at most one request arrives. The reason behind this is twofold. First, we aim to bound the number of possible scenarios, since this facilitates the computations. Second, if $\Delta t$ is small, it is not very likely that two or more requests arrive in the same time period. The probability that such a scenario in rural regions occurs is relatively small, and we do not consider this.

Consider state $s = (x, b, f, z, Z, D)$. We generate all possible outcomes of the evolutionary process described in Section 2.2.3, with the restrictions that $\sum_{i \in H} \omega^i \leq 1$ and $a^i_0 = 0$, $i \in N$. That is, the number of arriving calls is bounded by 1 and each ambulance keeps its position. Let this set of possible scenarios when sampling from state $s$ under these restrictions be denoted by $S(s)$ and

$$s^n = (x^n, b^n, f^n, z^n, Z^n, D^n) \in S(s)$$

denote the $n$-th scenario, where $1 \leq n \leq |S(s)|$. Moreover, $\mathbb{P}\{s' = s^n|s\}$ denotes the probability that scenario $n$ occurs. Due to the restriction on the number of requests that can happen at the same time, it holds that

$$\sum_{n=1}^{|S(s)|} \mathbb{P}\{s' = s^n|s\} < 1.$$ 

For the calculation of $\mathbb{P}\{s' = s^n|s\}$, we use a slightly different arrival process of requests, since we know that at most one request arrives. Before, at demand location $i$ exactly one request occurred with probability $p_i e^{-p_i}$, due to the fact that the number of arriving requests is Poisson distributed. However, for the calculation of the scenario probabilities, we assume that at location $i$ exactly one request occurs with probability $1 - e^{-p_i}$. That is, we add the probability of more than one incoming request to the probability of exactly one incoming request. In the next section, we calculate the probability that scenario $n$ occurs for each of the five state components, step by step.

2.3.1 Scenario Probabilities

1. The number of patients per demand location. We first consider $\mathbb{P}\{x' = x^n|s\}$ for scenario $n$. Since the arrival and completion process of requests is node-wise independent, it holds that

$$\mathbb{P}\{x' = x^n|s\} = \prod_{i=1}^{N} \mathbb{P}\{x'_i = x^n_i|s\}. \quad (2.12)$$

As in Section 2.2.3, we distinguish between $h \in H$ and $i \in \bar{H}$. We consider the case $h \in H$ first. The arrival process is defined by the occupied ambulances arriving
to \( h \). This number is given by \( D(h, 1) \). The number of patients in scenario \( n \) for which the treatment is not completed, denoted by \( G_h^n \) at \( h \), is Poisson binomially distributed. That is,

\[
\mathbb{P}\{G_h^n = g_h^n | s\} = \sum_{U \in U_{b_h}(g_h^n), j_1 \in U} \prod_{j_2 \in U} (1 - \rho_h^{j_1}) \rho_h^{j_2},
\]

where \( U_{b_h}(g_h^n) \) is the set of subsets of \( \{1, 2, \ldots, b_h\} \) with exactly \( g_h^n \) elements. Moreover, \( U \) is the complement of \( U^i \) in \( \{1, 2, \ldots, b_h\} \) and \( \rho_h^{j_1} \) is the probability that the \( j \)-th patient at \( h \) will have been treated at the next time step. If \( D(h, 1) = j \) patients arrive in one time step at \( h \), then the total number of patients at \( h \) in scenario \( n \) is in \( \{j, j+1, \ldots, j+x_h\} \). Moreover, given that \( j \) patients arrive and in scenario \( n \) we have \( x_h^n \) patients at \( h \), we observe that for \( x_h^n - j \) treatment is not completed. Now, we find

\[
\mathbb{P}\{x_i' = x_i^n | s\} = \sum_{j=0}^A \mathbb{1}_{\{D(h,1)=j\}} \mathbb{1}_{\{j \leq x_h^n \leq x_h^n+j\}} \mathbb{P}\{G_h^n = x_h^n - j | s\}.
\]

For \( i \in \mathcal{H} \), the arrival process of requests is not deterministic: a request arrives with probability \( 1 - e^{-p_i} \). The total number of patients for which the service on scene is finished at \( i \) is again Poisson binomially distributed. Therefore,

\[
\mathbb{P}\{x_i' = x_i^n | s\} = \begin{cases} 
(1 - e^{-p_i}) \times \\
\mathbb{P}\{G_i^n = b_i - (x_i - x_i^n + 1) | s\} + \\
e^{-p_i} \mathbb{P}\{G_i^n = b_i - (x_i - x_i^n) | s\} & \text{if } x_i - b_i \leq x_i^n, x_i^n \leq x_i + 1, \\
e^{-p_i} \mathbb{P}\{G_i^n = 0 | s\} & \text{if } x_i^n = x_i - b_i, \\
(1 - e^{-p_i}) \mathbb{P}\{G_i^n = b_i | s\} & \text{if } x_i^n = x_i + 1, \\
0 & \text{else,}
\end{cases}
\]

where \( b_i \) denotes the number of ambulances that are busy serving a patient, i.e., the number of patients that are treated by an ambulance. The first part of the sum above considers the situation in which a request arrives at \( i \), with probability \( 1 - e^{-p_i} \). Mind that this arriving request cannot be served immediately.

2. The number of ambulances either in phase 1, 2, or 4 per demand location. Now, we consider the transition probabilities for the second component. Similar to Equation (2.12), this probability can be written in product-form:

\[
\mathbb{P}\{y_i' = y_i^n | s, x_i^n\} = \prod_{i=1}^N \mathbb{P}\{y_i' = y_i^n | s, x_i^n\}
\]

for scenario \( n \). Note that \( y_i^n \) depends on \( x_i^n \). If in scenario \( n \), \( x_i - x_i^n \) treatments on scene at location \( i \) are finished, and these ambulances all leave for a hospital, we find that \( y_i^n = y_i - (x_i - x_i^n) \). Moreover, there can be multiple possibilities for \( y_i^n \) that correspond to \( x_i^n \). This is the case if for a particular location \( i \in \mathcal{H}, \)
we have multiple busy ambulances and at least one but not all of these need to go to a hospital. If, in scenario \( n \), no requests arrive at \( i \) and \( 0 < x_i - x_i^n < x_i \) ambulances finish service on scene, we do not know how many of these \( x_i - x_i^n \) ambulances need to transport a patient. Thus,

\[
y_i - \min\{z_i, x_i - x_i^n\} \leq y_i^n \leq y_i - \max\{0, (x_i - x_i^n) - (b_i - z_i)\}.
\]

Assume that \( x_i^n \) is given, and that in node \( i \) no hospital is located, i.e., \( i \in \mathcal{H} \). We make a distinction whether no or one extra request is considered at \( i \) in scenario \( n \). If no extra request is considered, then for \( x_i - x_i^n \) patients the treatment on scene ends. If an additional request is considered, then \( x_i - x_i^n + 1 \) ambulances finish their service at location \( i \). Let \( \mathbb{P}\{0|s, x_i^n\} \) denote the probability that \( x_i^n \) does not include an extra request and let \( \mathbb{P}\{1|s, x_i^n\} \) denote the probability that it does. Because we assume that no more than one request can arrive per time period, it holds that \( \mathbb{P}\{0|s, x_i^n\} + \mathbb{P}\{1|s, x_i^n\} = 1 \). We distinguish three cases:

1. If \( x_i^n = x_i - b_i \), all busy ambulances complete their treatment on scene and no request arrives. Hence, \( \mathbb{P}\{0|s, x_i^n\} = 1 \).

2. If \( x_i^n = x_i + 1 \), no ambulance completes its treatment on scene and one request arrives. Therefore, \( \mathbb{P}\{1|s, x_i^n\} = 1 \).

3. If \( x_i - b_i < x_i^n < x_i + 1 \), either no or one additional request is considered. Thus, \( \mathbb{P}\{0|s, x_i^n\} = e^{-p_i} \) and \( \mathbb{P}\{1|s, x_i^n\} = 1 - e^{-p_i} \).

Let \( \mathbb{P}\{y_i' = y_i^n|s, x_i^n, 0\} \) and \( \mathbb{P}\{y_i' = y_i^n|s, x_i^n, 1\} \) denote the probability that no and one extra request at \( i \) is considered in scenario \( n \), respectively. Then,

\[
\mathbb{P}\{y_i' = y_i^n|s, x_i^n\} = \mathbb{P}\{y_i' = y_i^n|s, x_i^n, 0\}\mathbb{P}\{0|s, x_i^n\} + \mathbb{P}\{y_i' = y_i^n|s, x_i^n, 1\}\mathbb{P}\{1|s, x_i^n\}.
\]

(2.13)

First, we determine \( \mathbb{P}\{y_i' = y_i^n|s, x_i^n, 0\} \) in order to compute the left-hand side of Equation (2.13). Of the \( b_i \) busy ambulances at location \( i \), \( x_i - x_i^n \) finish their service on scene. If \( y_i^n \) ambulances remain at \( i \), then \( y_i - y_i^n \) of the \( z_i \) ambulances that have to transport a patient to a hospital leave location \( i \). The remainder of the \( x_i - x_i^n \) ambulances that complete their treatment on scene (that is, \( (x_i - x_i^n) - (y_i - y_i^n) \) ambulances) finished serving patients that do not need transportation, of which there are \( b_i - z_i \). Hence,

\[
\mathbb{P}\{y_i' = y_i^n|s, x_i^n, 0\} = \frac{b_i - z_i}{(x_i - x_i^n) - (y_i - y_i^n)} \left( \frac{z_i}{y_i - y_i^n} \right),
\]

where we define \( \binom{K}{\kappa} = 0 \) if \( \kappa < 0 \) or \( \kappa > K \). Note that \( x_i - x_i^n \leq b_i \), so the denominator is always positive.
If we consider one extra request, then \( x_i - x_i^n + 1 \) ambulances finish their service on scene. Then, if \( x_i - x_i^n + 1 \leq b_i \), it holds that
\[
P\{y'_i = y_i^n | s, x_i^n, 1\} = \frac{b_i - z_i}{b_i} \left( \frac{y_i - y_i^n}{x_i - x_i^n + 1} \right).
\]
If \( x_i - x_i^n + 1 > b_i \), we define \( P\{y'_i = y_i^n | s, x_i^n, 1\} \) to be 0. However, if this is the case, \( P\{1 | s, x_i^n\} = 0 \), so the second term in Equation (2.13) vanishes.

Note that for \( h \in H \), it holds that \( y^n_h \geq y_h \), since we restrict ourselves to the action in which none of the idle ambulances leave for a neighbour. However, \( D(h, 1) \) occupied ambulances arrive at \( h \) in the next time step. Therefore, \( y^n_h = y_h + D(h, 1) \) for each scenario \( s^n \in S(s) \). Hence,
\[
P\{y_h = y_h^n | s\} = \begin{cases} 1 & \text{if } y_h^n = y_h + D(h, 1), \\ 0 & \text{else.} \end{cases}
\]

Using Equation (2.12), we can now compute the transition probabilities corresponding to the second component.

3. The number of busy ambulances per demand location required to transport patients. We now compute
\[
P\{z' = z^n | s, x^n, y^n\} = \prod_{i=1}^{N} P\{z'_i = z^n | s, x_i^n, y_i^n\}.
\]

We know that \( P\{z'_i = 0 | s, x_i^n, y_i^n\} = 1 \), so we consider the case \( i \in \bar{H} \). Remember that \( d_i^n \) denotes the number of ambulances that start a new treatment on scene at \( i \) in scenario \( n \). As before, we make a distinction whether no or one extra request is considered at \( i \) in scenario \( n \). Let \( d_i^n(u), u = 0, 1 \), denote the same quantity, but conditioned on the number of additional requests considered. Then, similar to what was done in the previous section, we find that
\[
d_i^n(0) = \min\{(x_i - x_i^n) - (y_i - y_i^n) + f_i, x_i - b_i\},
\]
using Equation (2.4). The first part corresponds to the number of ambulances that possibly can start a new treatment, while the second part equals the number of requests not treated by an ambulance at the moment. The number of ambulances that complete their treatment on scene is \( x_i - x_i^n \), of which \( y_i - y_i^n \) leave for a hospital, occupied by a patient. Besides, all idle ambulances at \( i \) can start a new service. Moreover, there are no incoming requests, so the treatment of \( x_i - b_i \) patients could be started if there were enough ambulances. These \( d_i^n(0) \) ambulances all make a diagnosis whether the patients they are serving need transportation. Therefore,
\[
z_i - (y_i - y_i^n) \leq z_i^n \leq z_i - (y_i - y_i^n) + d_i^n.
\]
The number of patients for which it is decided that they need transportation is binomially distributed on \(d^n_i(0)\) trials. Remember that the probability of transportation is \(r\). Note that

\[
d^n_i(1) = \min\{ (x_i - x^n_i + 1) - (y_i - y^n_i) + f_i, 1 + x_i - b_i \} = d^n_i(0) + 1,
\]
again by using Equation (2.4). Then, for \(u = 0, 1, \ldots\):

\[
\mathbb{P}\{ z'_i = z^n_i | s, x^n_i, y^n_i, u \} = \begin{cases} 
\binom{d^n_i(u)}{j} r^j(1 - r)^{d^n_i(u) - j} & \text{if } z^n_i = z_i - (y_i - y^n_i) + j, \\
0 & \text{else,}
\end{cases}
\]

and

\[
\mathbb{P}\{ z'_i = z^n_i | s, x^n_i, y^n_i \} = \sum_{u=0}^{1} \mathbb{P}\{ z'_i = z^n_i | s, x^n_i, y^n_i, u \} \mathbb{P}\{ u | s, x^n_i \}.
\]

4. The elapsed service time of ambulances in phases 2 and 4. Computing \(\mathbb{P}\{ Z' = Z^n | s, x^n, y^n \} \) requires more work. Let \(h \in \mathcal{H}\) and denote the \(h\)-th row of \(Z\) by \(Z(h)\). Then

\[
\mathbb{P}\{ Z' = Z^n | s, x^n, y^n \} = \prod_{h \in \mathcal{H}} \mathbb{P}\{ Z'(h) = Z^n(h) | s, x^n_h, y^n_h \}.
\]

We assume that \(Z(h)\) is always sorted in non-increasing order. That is, the first \(b_h\) entries of \(Z(h)\) denote the elapsed service times at \(h\), and the remainder of the row equals \(-1\). However, if an ambulance ends the treatment of a patient, its past service time is excluded from \(Z(h)\). In other words, there is an extra \(-1\). But since we assume \(Z'\) is sorted in non-increasing order, this \(-1\) is placed among the last entries of \(Z'(h)\). Thus, \(Z'(h, j)\) does possibly not correspond to the same ambulance to which \(Z(h, j)\) corresponds.

Let \(\hat{Z}(h) \sim Z'(h)\), where the notation ‘\(\sim\)’ means that if we sort \(\hat{Z}(h)\) in non-increasing order, it equals \(Z'(h)\). One can check that ‘\(\sim\)’ indeed defines an equivalence relation. Moreover, if \(\hat{Z}(h) \sim Z'(h)\), it holds that

\[
\mathbb{P}\{ \hat{Z}(h) = Z^n(h) | s, x^n_h, y^n_h \} = \mathbb{P}\{ Z'(h) = Z^n(h) | s, x^n_h, y^n_h \}.
\]

We divide \(Z^n(h)\) in three parts. The first part consists of the first \(b_h\) entries corresponding to the ambulances that are treating a patient at \(h\) in \(s\). The probability that ambulance \(j\) finishes its treatment is \(\rho^j_h\) defined in Equation (2.3), where \(1 \leq j \leq b_h\). Then, we find that

\[
\mathbb{P}\{ \hat{Z}(h, j) = Z^n(h, j) | s \} = \begin{cases} 
\rho^j_h & \text{if } Z^n(h, j) = -1, \\
1 - \rho^j_h & \text{if } Z^n(h, j) = Z(h, j) + 1, \\
0 & \text{else.}
\end{cases}
\]

Note that we do not condition on \(x^n\) and \(y^n\) here since these determine how many ambulances end their treatments. Moreover, \(y^n_h - y_h\) occupied ambulances arrive
at \( h \). Hence, \( x^n_h - (y^n_h - y_h) \) of the \( b_h \) ambulances remain busy, while the rest finishes its treatment. Let \((Z(h,j))_{j \leq b_h}\) denote the first \( b_h \) entries of \( Z(h) \). We denote the number of patients at \( h \) for which the treatment is \textit{not} completed in scenario \( n \) by \( G^n_h \). If \( \sum_{j=1}^{b_h} \mathbb{I}\{Z^n(h,j) > 0\} = x^n_h - (y^n_h - y_h) \), then

\[
\mathbb{P}\left\{ (\hat{Z}(h,j))_{j \leq b_h} = (Z^n(h,j))_{j \leq b_h} \mid s, x^n_h, y^n_h \right\} = \frac{\prod_{j=1}^{b_h} \mathbb{P}\{\hat{Z}(h,j) = Z^n(h,j)\mid s\}}{\mathbb{P}\{G^n_h = x^n_h - (y^n_h - y_h)\}},
\]

and 0 if this is not the case. The second part corresponds to the \( D(h,1) \) arriving ambulances at \( h \). Therefore, \( \hat{Z}(h,j) = 0 \) for \( b_h + 1 \leq j \leq b_h + D(h,1) \). Hence,

\[
\mathbb{P}\{\hat{Z}(h,j) = Z^n(h,j)\mid s, x^n_h, y^n_h \} = \begin{cases} 1 & \text{if } Z^n(h,j) = 0, \\ 0 & \text{else.} \end{cases} \quad (2.14)
\]

In the last part, \( b_h + D(h,1) + 1 \leq j \leq A \), and thus \( \hat{Z}(h,j) = -1 \). Therefore,

\[
\mathbb{P}\{\hat{Z}(h,j) = Z^n(h,j)\mid s, x^n_h, y^n_h \} = \begin{cases} 1 & \text{if } Z^n(h,j) = -1, \\ 0 & \text{else.} \end{cases} \quad (2.15)
\]

For \( i \in \bar{H} \), computing \( \mathbb{P}\{\hat{Z}(i,j) = Z^n(i,j)\mid s, x^n_i, y^n_i \} \) differs slightly. The first part, for \( j \leq b_i \), is similar. For the second part, Equation (2.14) holds for \( b_i + 1 \leq j \leq b_i + d^n_i \), since \( d^n_i \) ambulances start a new treatment. Consequently, Equation (2.15) holds for \( b_i + d^n_i + 1 \leq j \leq A \).

5. Destinations and remaining driving times of ambulances in phase

3. To compute \( \mathbb{P}\{D' = D^n\mid s, x^n, y^n \} \), we again consider \( h \in \mathcal{H} \). All ambulances that were already driving to \( h \) have progressed one unit distance at the next time, which is the length of one edge. Hence,

\[
\mathbb{P}\{D'(h) = D^n(h)\mid s, x^n_h, y^n_h \} = \prod_{h \in \mathcal{H}} \prod_{t=1}^{L} \mathbb{P}\{D'(h,t) = D^n(h,t)\mid s, x^n_h, y^n_h \}.
\]

Remember that for \( i \in \bar{H} \), \( y_i - y^n_i \) ambulances leave for a hospital, all to hospital \( h \) for which \( i \in H^*(h) \). Therefore,

\[
D^n(h,t) = D(h,t + 1) + \sum_{i \in \mathcal{H}} (y_i - y^n_i) \mathbb{I}\{i \in H^*(h)\} \mathbb{I}\{l(i,h) = t\},
\]

implies that

\[
\mathbb{P}\{D'(h,t) = D^n(h,t)\mid s, x^n_h, y^n_h \} = 1,
\]

and 0 if this is not the case. Now, we have described all ingredients to compute \( \mathbb{P}\{s' = s^n\mid s\} \), the probability that scenario \( s^n = (x^n, y^n, z^n, Z^n, D^n) \) occurs.
2.3.2 Response Time Expectations

Up to now, we assumed that the action we take is the action in which none of the ambulances moves, except for those transporting a patient to a hospital. In this section, we drop this assumption and we combine each scenario with each feasible action, which results in a potential new state. However, the expected response time to the additional request in this potential state is of more interest to us than the potential state itself. We consider the following ambulances as eligible for responding to this patient:

(I) The nearest idle unassigned ambulance,

(II) The nearest busy ambulance(s) transferring a patient at a hospital,

(III) The nearest busy ambulance(s) treating on scene, of which it is known that it is not required to transport a patient.

Note that we do not consider ambulances that are transporting patients. These ambulances will be busy for a deterministic remaining driving time and a stochastic treatment time at the hospital. Therefore, it probably takes ample time before they can be assigned to another incident. For the same reason, we assume ambulances treating on scene that know that they have to transport the patient they are serving, are not eligible. Formally, let $s = (x, y, z, Z, D)$ be our current state, where $y = f + b$, and let $a^s \in \mathcal{F}(s)$ be the action we take. Consider scenario $s^n$. We define $\bar{s}(s^n, a^s)$ as the state in which action $a^s$ has been carried out in scenario $s^n$. Moreover,

$$\bar{s}(s^n, a^s) = (\bar{x}(s^n, a^s), \bar{y}(s^n, a^s), \bar{z}(s^n, a^s), \bar{Z}(s^n, a^s), \bar{D}(s^n, a^s)),$$

and we omit the dependence on $s^n$ and $a^s$ in the remainder. The state components are defined as follows:

$$\bar{x}_i = b^n_i + 1_{\{x^n_i = x_i + 1\}}, \quad i \in \mathcal{N},$$

i.e., in $\bar{x}$ only the patients that are being treated on scene and the additional waiting patient are considered. That is, we do not consider the waiting patients that were already present in $s$. Moreover,

$$\bar{y}_i = y^n_i + a^s_i - \max\{0, \min_{a^s \in \mathcal{F}(s)} \alpha^s_i\}, \quad i \in \mathcal{N},$$

in other words, only the eligible ambulances mentioned are considered. Note that if $\min_{a^s \in \mathcal{F}(s)} \alpha^s_i > 0$, each feasible action dispatches at least one ambulance to location $i$. Hence, location $i$ is on the shortest path to a node where a patient waits. We do not consider ambulances traveling to waiting patients as eligible ones and therefore exclude them from $\bar{y}$. Furthermore, $\bar{z} = z^n$, $\bar{Z} = Z^n$, and $\bar{D} = D^n$. These state components do not depend on the action taken. We now compute the expected response time for the additional patient in $\bar{s}$ from the eligible ambulances. All eligible ambulances are observed in $\bar{y}$. Denote these response times from the
ambulances defined in (I), (II), and (III) by $R^{(i)}(\bar{s})$, where $i \in \{I, II, III\}$. We compute $\mathbb{E}\{R^{(i)}(\bar{s})\}$ for the additional patient in $\bar{s}$ as follows:

**I.** The determination of $\mathbb{E}\{R^{(I)}(\bar{s})\}$ is easy, since there is no randomness involved in its computation. The response time for the patient waiting from the nearest idle unassigned ambulance is just the travel time from the current location of the ambulance to the waiting patient. Assume that the additional patient in scenario $n$ is at location $i$. Then,

$$\mathbb{E}\{R^{(I)}(\bar{s})\} = \min_{j : \bar{y}_j > \bar{x}_j} l(j,i),$$

using that if for location $j$ it holds that $\bar{y}_j > \bar{x}_j$, we have an idle unassigned ambulance at $j$.

**II.** Now we compute $\mathbb{E}\{R^{(II)}(\bar{s})\}$. Of all hospitals, we consider the nearest hospital with at least one busy ambulance. Possibly, there are more busy ambulances at this hospital. The expected response time from one of these ambulances consists of two parts: the expected time until at least one ambulance finishes its treatment and a deterministic travel time from the hospital to the additional patient. Assume that the patient waiting in $\bar{s}$ is at location $i$. Moreover, suppose that hospital $h$ is the nearest hospital with at least one busy ambulance. Assume that $\bar{b}_h$ ambulances are busy at $h$. The elapsed service time of ambulance $j$ at hospital $h$ is given by $\bar{Z}(h,j)$. For each of these $\bar{b}_h$ ambulances, we can compute the probabilities that they finish their treatment in exactly $t$ time units from now. That is, we compute

$$\rho^j_h(t) = \mathbb{P}\{B^j_h = \bar{Z}(h,j) + t | B^j_h > \bar{Z}(h,j)\}, \quad t \geq 1.$$\hspace{1cm} (2.16)

Now, define $T(h)$ to be the number of time steps it takes for at least one busy ambulance at $h$ to complete its service. Then,

$$\mathbb{P}\{T(h) = 1|\bar{Z}(h)\} = 1 - \prod_{j=1}^{\bar{b}_h} \left( 1 - \rho^j_h(1) \right),$$

which is the probability that at least one ambulance ends its treatment after exactly one time unit from now. We can generalize this to $t$ time units as follows:

$$\mathbb{P}\{T(h) = t|\bar{Z}(h)\} = \left( 1 - \prod_{j=1}^{\bar{b}_h} \left( 1 - \rho^j_h(t) \right) \right) \left( 1 - \sum_{\tau=1}^{t-1} \mathbb{P}\{T(h) = \tau|\bar{Z}(h)\} \right),$$\hspace{1cm} (2.17)

where $t \geq 1$ and the last part corresponds to the probability that none of the busy ambulances at $h$ finished its treatment before time $t$. Now, we compute

$$\mathbb{E}\{T(h)|\bar{Z}(h)\} = \sum_{t=1}^{\infty} t \mathbb{P}\{T(h) = t|\bar{Z}(h)\},$$\hspace{1cm} (2.17)
which is the expected time until an ambulance at \( h \) ends its service if the system is in state \( \tilde{s} \). The expected response time to the additional patient from ambulance(\( s \)) at the nearest hospital with at least one busy ambulance is given by

\[
E\{R^{(II)}(\tilde{s})\} = E\{T(h) | \tilde{Z}(h)\} + l(h, i),
\]

(2.18)

where we assume that the additional patient in scenario \( n \) is at location \( i \) and \( h = \arg \min \{ l(i, h) : \hat{b}_h > 0, h \in H \} \), i.e., the nearest hospital with at least one busy ambulance. If no such \( h \) exists, we define \( E\{R^{(II)}(\tilde{s})\} = \infty \).

(III). The term \( E\{R^{(III)}(\tilde{s})\} \) consists of two parts as well: the expected time until an ambulance finishes its treatment and a deterministic travel time. The computation of \( E\{R^{(III)}(\tilde{s})\} \) is similar to \( E\{R^{(II)}(\tilde{s})\} \), and we assume \( E\{R^{(III)}(\tilde{s})\} = \infty \) if there is no ambulance not required to transport, while treating a patient on scene.

Given \( \tilde{s} \), we compute the shortest expected response time to the additional patient in \( s^n \) that is possible from the eligible ambulances. Let this quantity be defined by \( E\{R(\tilde{s})\} \): it reassembles the shortest response time possible if each busy ambulance would finish its treatment, either on scene or at the hospital, now. This quantity is given by

\[
E\{R(\tilde{s})\} = \min_{\iota \in \{I, II, III\}} E\{R^{(\iota)}(\tilde{s})\}
\]

and it equals zero if and only if at the location of the additional patient there is an idle unassigned ambulance as well.

### 2.3.3 Action Selection

Consider state \( s \) and \( F(s) \), which is the set of feasible actions as computed by solving one of the two assignment problems mentioned in Section 2.2.2. For each feasible action, we compute the penalty of the weighted average shortest expected response time to an arriving request, using the penalty function \( \Phi \) introduced in Section 2.2.4. This quantity serves as the measure for the effect the action has on the EMS system. For action \( a^s \), we denote this quantity by \( V(a^s) \) and we compute it by

\[
V(a^s) = \sum_{n=1}^{\lfloor S(s) \rfloor} \Phi \left( E\left\{ R(\tilde{s}(s^n, a^s)) \right\} \right) P\{ s' = s^n | s \},
\]

(2.19)

where \( E\{R(\tilde{s})\} \) and \( P\{ s' = s^n | s \} \) are described in Sections 2.3.2 and 2.3.1, respectively. Note that \( E\{R(\tilde{s})\} \) is not necessarily integer, so for this reason \( \Phi \) needs to be continuous. We compute Equation (2.19) for each action in \( F(s) \). Then, we select the action \( a^s \in F(s) \) for which

\[
a^s = \arg \min_{\alpha \in F(s)} V(\alpha^s).
\]

That is, the action that minimizes the weighted average shortest expected response time to an arriving request in the upcoming time period, is taken.
2.3.4 Theoretical Weaknesses

We end this section with the discussion of two theoretical weaknesses of the heuristic described above: situations in which the heuristic might perform poorly. A first limitation of the method is that it considers only the nearest of the eligible ambulances: an ambulance driving from one town to another is not observed by the heuristic if in both towns an ambulance is present. A solution in which this ambulance is observed might result in a better policy. However, this only plays a role if the probability of having multiple busy ambulances per town is large, which is typically not the case in the rural regions we observe.

Moreover, another possible weakness of this heuristic is that only an ambulance configuration in the ‘neighborhood’ of the current configuration can be attained in the next time step. This is a consequence of the fact that ambulances cannot traverse more than one edge per time unit. Therefore, the best action selected might not lead closer to the global optimal ambulance configuration. This is illustrated in the following small example.

Consider a chain with five equidistant nodes, where nodes 1 and 5 represent points of relatively high demand. The demand in the middle nodes 2, 3 and 4 is very low, as is typically the case in rural regions. Moreover, assume that the demand in node 5 is significantly higher than the demand in node 1. We use the penalty function of Figure 2.3a with $T_{max} = 1$ and we assume we have only one ambulance. The global optimal solution is to locate the ambulance at node 4, since it covers nodes 3, 4 and the high demand of node 5. However, if the ambulance is at node 1, the ambulance ends up in node 2. This is a consequence of the fact that the action of traversing the edge between nodes 2 and 3 is classified as a bad action, because if the ambulance is in node 3, then neither node 1 nor node 5 is covered. That is, instead of the global optimal configuration, a local optimal configuration is attained. However, instead of a weakness one can also interpret this as a strength, because attaining a local optimal configuration involves less driving. To investigate this, we compare the heuristic to a policy that focuses on attaining the global optimal configuration: the compliance table policy.

Two other assumptions that could impact the performance of the algorithm are the limitation of the scenarios with only one additional incident and the one-step lookahead. Relaxing these assumptions seriously increases the computation time and the question arises whether this is beneficial. This is probably not the case, since we focus on rural regions and as a consequence, the probability that two consecutive requests arrive in a short period of time, is relatively small. Results in Section 2.4.2 below, in which we compare the heuristic with the optimal policy for a small example, show indeed that the heuristic performs near-optimal for the performance indicator related to the chosen penalty function.

2.4 Numerical Results

The heuristic described in the previous section computes for each state an action in which the expected penalty is minimized. We call the policy obtained by per-
forming the heuristic the **heuristic policy**. We compare it to a different policy: the compliance table policy, which we will explain in the next subsection.

## 2.4.1 Compliance Tables

Compliance table policies are commonly used in practice for dynamic ambulance management (see Alanis et al. (2013), Gendreau et al. (2006)). Each row in a compliance table shows, for a given number of idle ambulances, the desired locations for these ambulances. If these ambulances are at their desired location, the system is **in compliance**. The number of idle ambulances changes when a request arrives or when an ambulance becomes idle again. Then, each idle ambulance is assigned to a possible new location. That is, in state $s$ we first solve an assignment problem to assign ambulances to requests. After that, we solve a second assignment problem for the unassigned ambulances and desired locations. In our computations, we used LBAP for both assignment problems. Moreover, we assume that each ambulance immediately starts driving to its desired location.

We want to compare the heuristic described in Section 2.3 to a good compliance table with respect to the chosen penalty function. After all, for different penalty functions, compliance tables may differ. We assume that no more than one ambulance is deployed at a single location, because our setting is a rural region with a small number of ambulances. The arrival rate of incidents is low, and thus it is very unlikely that a second incident occurs just after the first in a certain area. We generate compliance tables by solving a static optimization problem for each level independently: the $p$-median problem.

In the $p$-median problem, which was formulated as an integer linear program by ReVelle and Swain (1970), one aims to find the location of a fixed number of facilities so as to minimize the weighted average distance. In the context of dynamic ambulance management, this translates to finding the location of the idle ambulances in such a way that the weighted sum over each node of the distance from the node to the nearest ambulance is minimized. Remember that $l(i, j)$ is the length of the shortest path between nodes $i$ and $j$, and $p$ is the parameter of the Poisson distribution that models the number of arriving requests. However, we do not use the shortest-path lengths itself, but the penalties corresponding to these to incorporate the penalty function of interest. The objective function is as follows:

$$\text{Minimize } \sum_{i \in N} \sum_{j \in N} p_i \Phi(l(i, j)) Y_{ij},$$  \hspace{1cm} (2.20)

where $Y_{ij}$ is a binary decision variable: $Y_{ij} = 1$ if and only if a request at node $i$ is served by an ambulance at node $j$, i.e., if the ambulance at $j$ is the closest ambulance to node $i$. In addition, we introduce a binary decision variable $X_j$ which equals one if an ambulance is placed at location $j$. Assume that there are $F$ idle ambulances. Thus, we compute the $F$-th row of the compliance table. We
2.4 Numerical Results

Figure 2.4: EMS region of Flevoland: spatial distribution of requests (A), simplified graph model (B), and extended graph model (C).

minimize Equation (2.20) under the following constraints:

\[
\begin{align*}
\sum_{j \in \mathcal{N}} Y_{ij} &= 1 \quad i \in \mathcal{N} \\
\sum_{j \in \mathcal{N}} X_j &= F \\
Y_{ij} &\leq X_j \quad i, j \in \mathcal{N} \\
Y_{ij}, X_j &\in \{0, 1\}.
\end{align*}
\]

The first constraint states that each request has exactly one ambulance that is nearest. We need to find the desired locations of \( F \) ambulances, which is given by the second constraint. The third constraint induces that an ambulance at \( j \) can serve a request at node \( i \) only if \( j \) is a desired location. For each value of \( F \), \( 1 \leq F \leq A \), we solve this \( p \)-median problem.

Note that there is no cohesion between the compliance table levels by applying this procedure. After all, the problem is solved for each level independently. Therefore, compliance table \( k \) aims for the optimal global configuration with \( k \) available ambulances, not a near-optimal one that can be attained faster in order to be in compliance earlier. However, by the same reasoning as before, the time between consecutive incidents is supposed to be large. As a consequence, it is justified to assume that there is enough time to attain the optimal configuration for this number of available ambulances before a next incident occurs. In short, given that no more than one ambulance is placed at a single location and each level is computed independent of each other, this procedure computes the optimal compliance table.
To gain insight into the performance of both the heuristic and the compliance table policy, we first compare them to the optimal policy in an illustrative example. We apply these three policies to an EMS system belonging to a rural region in the Netherlands: Flevoland. A map of this region, as well as the spatial distribution of requests, is displayed in Figure 2.4a. We set $\Delta t$ equal to 15 minutes and model the region by the graph in Figure 2.4b with 11 nodes and 15 edges. On average, there are 28.6 requests per day in our problem instance. There are six nodes with a non-zero arrival parameter, which varies between 1.2 and 15.1 requests per day. We consider an instance with four ambulances and we assume that none of the patients has to be transported to a hospital in our example. The treatment time on scene follows a geometric distribution with parameter 0.3. This results in a mean treatment time on scene of 50 minutes. These two simplifications greatly reduce the size of the state space, as now a state is described by the first two components only: $(x, y)$. In order to compute the optimal policy, we truncate the state space by assuming that the maximum number of requests, denoted by $\bar{X}$, is five: $\sum_{i=1}^{N} x_i \leq \bar{X} = 5$. This results in a state space of 630,630 18-dimensional states, computed by

$$\sum_{i=0}^{\bar{X}} \binom{N'}{i} \left( \binom{A+N-1}{N-1} \right),$$

in which there are $N' \leq N$ nodes with a non-zero arrival parameter, $A$ ambulances and $N$ nodes in total. Here, $\bar{X} = 5$, $N' = 6$, $A = 4$ and $N = 12$.

We model the problem as an MDP for the linear penalty function $\Phi(t) = t$, and solve it using Value Iteration (c.f. Puterman (1994)). We use LBAP to compute the set of feasible actions. The average size of the set of feasible actions is 1.9 actions. There are many states in which we only allow one action, namely the states with 4 or 5 requests in total. The maximum number of feasible actions in a state is 321, which obviously was a state without any request. The compliance table, as computed by solving the $p$-median problem, is displayed in Table 2.4.

We simulate this table, the optimal policy and the heuristic policy for one million time steps. Results on late arrivals, response times and driving ambulances, as well as their 95% confidence bounds, are displayed in Table 2.5. The fraction of late arrivals represents the fraction of requests for which a maximum allowed response time of 15 minutes (1 time unit) is exceeded. The mean response time is expressed in time units. In the computation of the mean number of driving

<table>
<thead>
<tr>
<th>Level</th>
<th>Compliance Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1-2</td>
</tr>
<tr>
<td>3</td>
<td>1-2-9</td>
</tr>
<tr>
<td>4</td>
<td>1-2-4-6</td>
</tr>
</tbody>
</table>

Table 2.4: Compliance table of the simplified example.
Table 2.5: Results for the simplified example.

<table>
<thead>
<tr>
<th>Policy</th>
<th>Performance Statistics</th>
<th>Mean</th>
<th>95%-CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal</td>
<td>Fraction late arrivals</td>
<td>1.68%</td>
<td>[1.55%, 1.65%]</td>
</tr>
<tr>
<td></td>
<td>Mean response time (time units)</td>
<td>0.0587</td>
<td>[0.0568, 0.0607]</td>
</tr>
<tr>
<td></td>
<td>Mean no. driving ambulances</td>
<td>0.6280</td>
<td>[0.6259, 0.6301]</td>
</tr>
<tr>
<td>Heuristic</td>
<td>Fraction late arrivals</td>
<td>1.95%</td>
<td>[1.90%, 2.00%]</td>
</tr>
<tr>
<td></td>
<td>Mean response time (time units)</td>
<td>0.0390</td>
<td>[0.0371, 0.0609]</td>
</tr>
<tr>
<td></td>
<td>Mean no. driving ambulances</td>
<td>0.7232</td>
<td>[0.7213, 0.7251]</td>
</tr>
<tr>
<td>Compliance Table</td>
<td>Fraction late arrivals</td>
<td>2.22%</td>
<td>[2.16%, 2.27%]</td>
</tr>
<tr>
<td></td>
<td>Mean response time (time units)</td>
<td>0.0630</td>
<td>[0.0611, 0.0649]</td>
</tr>
<tr>
<td></td>
<td>Mean no. driving ambulances</td>
<td>0.9305</td>
<td>[0.9281, 0.9328]</td>
</tr>
</tbody>
</table>

As expected, the optimal policy outperforms the other two policies on the performance measure related to the penalty function, although the differences between the mean response time induced by the optimal and heuristic policy are really close and their 95% confidence bounds overlap almost entirely. As a consequence, on this performance criterion the heuristic policy is a near-optimal policy. This shows that the two main assumptions stated at the end of Section 2.3.4, namely the limitation to scenarios with only one additional request and the one-step lookahead, have a very small impact on the performance. Relaxing these assumptions will seriously increase the computation time while there is little room for improvement. The optimal policy performs better on the two other performance indicators as well. It is also worth noting that the performance gap between the optimal and heuristic policy is smaller than the gap between the heuristic and the compliance table policy for all performance measures.

If we compare the results of the heuristic and the compliance table policy in Table 2.5, we observe that the heuristic policy outperforms the compliance table on any of the three performance criteria. The difference on mean number of driving ambulances is explained by the fact that there is a drift to node 1 in the compliance table, because node 1 has the highest call arrival rate. Together with the fact that in this node a hospital is present, many ambulances become idle again here. The heuristic takes this into account by considering ambulances transferring a patient at a hospital as eligible ones as well. In contrast, the compliance table of Table 2.4 sends an ambulance from elsewhere to node 1 each time the ambulance present in node 1 is dispatched, which happens relatively much due to the high arrival rate. This results in a relatively large amount of driving.

2.4.3 Case Study 2

We apply both the heuristic policy and the compliance table policy to a more realistic model of the EMS region of Flevoland. We set $\Delta t$ equal to 5 minutes, and we model the region by the graph in Figure 2.4c, with 57 nodes and 74 edges. This
Table 2.6: Compliance tables.

<table>
<thead>
<tr>
<th>Level</th>
<th>Compliance Tables</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equation (2.7)</td>
<td>Equation (2.8)</td>
<td>Equation (2.9)</td>
</tr>
<tr>
<td>1</td>
<td>29</td>
<td>51</td>
<td>51</td>
</tr>
<tr>
<td>2</td>
<td>1-42</td>
<td>28-51</td>
<td>28-51</td>
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<tr>
<td>3</td>
<td>1-11-22</td>
<td>16-22-28</td>
<td>22-28-43</td>
</tr>
<tr>
<td>4</td>
<td>1-2-14-22</td>
<td>2-16-21-28</td>
<td>1-22-31-43</td>
</tr>
<tr>
<td>5</td>
<td>1-2-12-14-22</td>
<td>1-2-12-16-21</td>
<td>1-12-14-22-43</td>
</tr>
<tr>
<td>6</td>
<td>1-2-12-14-17-22</td>
<td>1-2-4-12-18-24</td>
<td>1-2-12-13-22-26</td>
</tr>
<tr>
<td>7</td>
<td>1-2-12-13-14-17-22</td>
<td>1-2-4-7-12-18-24</td>
<td>1-2-12-13-18-45-49</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Equation (2.10)</th>
<th></th>
<th>Equation (2.11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>29</td>
<td></td>
<td>29</td>
</tr>
<tr>
<td>2</td>
<td>28-51</td>
<td></td>
<td>28-51</td>
</tr>
<tr>
<td>3</td>
<td>16-21-28</td>
<td></td>
<td>22-28-43</td>
</tr>
<tr>
<td>4</td>
<td>2-22-28-51</td>
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<td>1-12-22-43</td>
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<tr>
<td>5</td>
<td>1-2-12-16-22</td>
<td></td>
<td>1-2-12-14-22</td>
</tr>
<tr>
<td>6</td>
<td>1-2-6-12-22-37</td>
<td></td>
<td>1-2-12-14-17-22</td>
</tr>
<tr>
<td>7</td>
<td>1-2-6-7-12-16-22</td>
<td></td>
<td>1-2-12-13-14-17-22</td>
</tr>
</tbody>
</table>

The time of 5 minutes corresponds to a road distance of approximately 5 kilometers in the towns and to 8 kilometers in the rural areas. There are two hospitals in the region, one in the city in the Southwest and one in the western city in the middle. We use historical data to estimate the several distributions needed. The node-dependent arrival parameter of requests varies between 0.12 and 4.3 requests per day. On average, there are 24.2 requests per day. For the on-scene time we estimate a geometric distribution with a mean of approximately 10 minutes, and a standard deviation of 7 minutes. The hospital treatment time follows a Discrete Weibull distribution. The mass-function of the Discrete Weibull distribution with parameters $\mu$ and $k$ is given by

$$P\{X = x\} = (1 - \mu)^{*k} - (1 - \mu)^{*(x+1)k}, \ x = 0, 1, 2, \ldots,$$

and is treated in detail by Rinne (2008). Here, $\mu = 0.1$ and $k = 2$, which results in a mean treatment time at the hospital of approximately 16 minutes and a standard deviation of 7.3 minutes. Moreover, 75% of the patients needs to visit a hospital, so $r = 0.75$. We consider cases with four ambulances and with seven ambulances, the latter being more realistic for this region.

We compute compliance tables for the five different penalty functions considered in Equations (2.7)–(2.11) in Section 2.2.4, where we take $\beta = 10$, $\gamma = 200$ and $T_{max} = 3$ time units (15 minutes). These functions are displayed in Figure 2.3 as
well. The computed compliance tables are displayed in Table 2.6. Note that for
Equation (2.7) and Equation (2.8), computing the compliance tables is equivalent
to solving a classical $p$-median problems and a MCLP-problems, respectively.

We use LBAP to compute the set of feasible actions. As was stated at the end
of Section 2.2.2, we incorporate the penalty function in this assignment problem.
We simulate 100,000 time steps and observe from the simulations values for four
different performance indicators, and their 95% confidence bounds. Results are
displayed in Tables 2.7 and 2.8.

Observing these tables, one can make several interesting observations. In terms
of penalty, the penalty function minimizing the number of late arrivals of Equa-
tion (2.8) combined with $A = 4$ is the only penalty function for which the heuristic
policy performs worse than the compliance table policy. This is probably due to
the fact that the heuristic policy only considers ambulance configurations that
can be attained in one time step. As a consequence of the small differentiation
in penalty for several response times, many actions are classified as equally good.
This is also reflected in the fact that although Equation (2.8) focuses on minimizing
the fraction of late arrivals, it is dominated on this criterion by three out of
the four other penalty functions in the case with four ambulances. Specifically,
the penalty function of Equation (2.9), that hardly differs from the one in Equa-
tion (2.8), performs much better on the fraction of late arrivals for the heuristic
policy. These phenomena do not occur in the case with seven ambulances. After
all, the action set is much larger in this case. Besides, with seven ambulances there
are more opportunities to cover low demand points as well. Hence, there is more
diversity in the classification of actions.

In general, the heuristic policy performs better than the compliance table poli-
cies on the mean response time for each of the considered penalty functions and
cases, although the difference is not significant for the penalty function of Equa-
tion (2.7). This is explained by the fact that the heuristic focuses on the shortest
expected response time from the eligible ambulances. In contrast to minimization
of the fraction of late arrivals, the penalty function that focuses on minimizing
the average response time, performs best on that criterion for both policies. The
largest gap in terms of response times between the two policies is observed for the
penalty function of Equation (2.8), in favour of the heuristic policy.

Comparing the fraction of late arrivals and the mean response times for each
penalty function in the heuristic policy in Table 2.7, one may note that in the
majority of the cases a shorter mean response time leads to an increase of the
fraction of late arrivals. This is also the case for the compliance table policies both
in Table 2.7 and Table 2.8. This negative correlation is in contrast to what one
intuitively might expect. Note that this phenomenon is most clear in Table 2.7
in the compliance table policies for penalty functions (2.7) and (2.8). This is
explained by the following reason. Equation (2.7) locates the ambulances close to
the city centers of the two largest towns. As a consequence, some minor towns
can not be reached within 15 minutes. Since approximately 56% of the incidents
occurs in the two largest towns, especially in the city centers, this results in small
response times to the areas of high demand. However, the response times to the
areas of low demand are much larger, but this is only marginally noted in the mean.
<table>
<thead>
<tr>
<th>Penalty Function</th>
<th>Performance Statistics</th>
<th>Heuristic Policy Compliance Table</th>
<th>95%–CI</th>
<th>Mean</th>
<th>95%–CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation (2.7)</td>
<td></td>
<td></td>
<td>12.44%</td>
<td>[12.19%, 12.70%]</td>
<td>14.08%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.5309</td>
<td>[0.5273, 0.5345]</td>
<td>0.8260</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.5433</td>
<td>[1.5305, 1.5601]</td>
<td>1.5665</td>
</tr>
<tr>
<td>Fraction late arrivals</td>
<td>Mean 95%-CI</td>
<td>Mean 95%-CI</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equation (2.8)</td>
<td></td>
<td></td>
<td>10.64%</td>
<td>[10.47%, 10.81%]</td>
<td>7.78%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.5831</td>
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<td>0.9132</td>
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<td></td>
<td>0.1064</td>
<td>[0.1047, 0.1081]</td>
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<tr>
<td>Equation (2.9)</td>
<td></td>
<td></td>
<td>8.24%</td>
<td>[8.04%, 8.45%]</td>
<td>8.61%</td>
</tr>
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<td>0.6537</td>
<td>[0.6495, 0.6578]</td>
<td>0.9521</td>
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<td></td>
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<td>0.0881</td>
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<tr>
<td>Equation (2.10)</td>
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<td>7.59%</td>
<td>[7.33%, 7.82%]</td>
<td>8.67%</td>
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<td>0.1659</td>
<td>[0.1589, 0.1739]</td>
<td>0.1948</td>
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<tr>
<td>Equation (2.11)</td>
<td></td>
<td></td>
<td>7.70%</td>
<td>[7.55%, 7.84%]</td>
<td>8.67%</td>
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<td>0.7106</td>
<td>[0.7054, 0.7158]</td>
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<td>0.2758</td>
<td>[0.2680, 0.2835]</td>
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<td>Penalty Function</td>
<td>Performance Statistics</td>
<td>Heuristic Policy</td>
<td>Compliance Table</td>
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<tr>
<td></td>
<td></td>
<td>Mean 95%-CI</td>
<td>Mean 95%-CI</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equation (2.7)</td>
<td>Fraction late arrivals</td>
<td>1.96% [1.46%, 2.46%]</td>
<td>2.51% [2.05%, 2.96%]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean no. of driving ambulances</td>
<td>0.3944 [0.3749, 0.4140]</td>
<td>0.8564 [0.8313, 0.8815]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean penalty per time step</td>
<td>0.7192 [0.6798, 0.7586]</td>
<td>0.7447 [0.7147, 0.7747]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equation (2.8)</td>
<td>Fraction late arrivals</td>
<td>1.18% [0.82%, 1.55%]</td>
<td>1.91% [1.65%, 2.18%]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean no. of driving ambulances</td>
<td>0.4147 [0.4012, 0.4283]</td>
<td>0.9325 [0.9093, 0.9557]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean penalty per time step</td>
<td>0.0118 [0.0082, 0.0155]</td>
<td>0.0192 [0.0165, 0.0218]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equation (2.9)</td>
<td>Fraction late arrivals</td>
<td>1.21% [0.93%, 1.49%]</td>
<td>1.82% [1.55%, 2.09%]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean response time (minutes)</td>
<td>3.7744 [3.6364, 3.9125]</td>
<td>5.6210 [5.5400, 5.7020]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean no. of driving ambulances</td>
<td>0.4508 [0.4373, 0.4644]</td>
<td>1.5192 [1.4949, 1.5436]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean penalty per time step</td>
<td>0.0122 [0.0094, 0.0150]</td>
<td>0.0184 [0.0157, 0.0211]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equation (2.10)</td>
<td>Fraction late arrivals</td>
<td>1.14% [0.86%, 1.41%]</td>
<td>1.95% [1.59%, 2.31%]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean response time (minutes)</td>
<td>3.5967 [3.4657, 3.7278]</td>
<td>5.7025 [5.5840, 5.8210]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean no. of driving ambulances</td>
<td>0.4259 [0.4140, 0.4378]</td>
<td>1.1518 [1.1294, 1.1741]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean penalty per time step</td>
<td>0.0183 [0.0124, 0.0243]</td>
<td>0.0296 [0.0238, 0.0354]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equation (2.11)</td>
<td>Fraction late arrivals</td>
<td>1.15% [0.86%, 1.44%]</td>
<td>1.93% [1.68%, 2.19%]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean no. of driving ambulances</td>
<td>0.4348 [0.4213, 0.4482]</td>
<td>0.8972 [0.8777, 0.9167]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean penalty per time step</td>
<td>0.0384 [0.0303, 0.0465]</td>
<td>0.0576 [0.0513, 0.0640]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In contrast, in the compliance table corresponding to penalty function (2.8) places ambulances in such a way that the demand that be reached within 15 minutes is maximized. Therefore, ambulances are further away from the areas of high demand, yielding a larger mean response time. An exception to this phenomenon is the heuristic policy in the case with seven ambulances. However, even in this case, the penalty function of Equation (2.7) induces the smallest response time, but the largest fraction of late arrivals.

Another interesting point is the number of driving ambulances. For each penalty function and each case, the heuristic policy greatly outperforms the compliance table policy on this performance indicator. This is caused by the fact that using compliance tables, one aims to attain an ambulance configuration only taking the number of available ambulances into account. In contrast, since ambulances can only traverse at most one edge per time unit, the heuristic computes a good local configuration. As a consequence, less driving is involved in using the heuristic policy. Moreover, comparing Tables 2.7 and 2.8, an increase in number of ambulances gives rise to a decrease of driving ambulances for the heuristic policy. In contrast, in the compliance table policy, more ambulances induce more driving in general, the penalty function of Equation (2.11) being the only exception.

If we compare Tables 2.7 and 2.8, we observe larger differences in patient-based results in the case with four ambulances. For instance, the fractions of late arrivals for the penalty functions of Equations (2.8)–(2.11) in the case with seven ambulances are very close to each other. In contrast, these differences in the case with four ambulances are much larger. Hence, a small change in setting (e.g., penalty function) may result in a large change in performance in such a case. This underlines what was stated in Section 2.1: if one has access to only a small number of ambulances, one has to be more careful about how to relocate them.

Apart from the first penalty function in the case with four ambulances, the heuristic outperforms the compliance table policy on each of the performance indicators. Therefore, it seems that attaining a good local ambulance configuration that can be reached quickly, performs better than attaining the desired configuration of ambulances supplied by the compliance table, which serves as a global configuration for this number of available ambulances.

2.5 Concluding Remarks

In this chapter, we proposed a DAM model for rural regions to solve the ambulance relocation problem. The model was formulated as a discrete-time Markov decision process. At each time step a relocation policy specifies, for each ambulance that is not busy, whether to move the ambulance to an adjacent node. A policy is sought that minimizes a general penalty function which is nondecreasing in the response time to a request. The function can be constructed to match the performance objectives of the EMS system being studied. Computation of the optimal policy in realistic settings is impractical, because the MDP has a high-dimensional state space. To address this, we developed a one-step look-ahead heuristic that, at each time step, relocates ambulances in order to minimize the expected response time.
for a possible call arriving in the next time step. We concluded this chapter with a numerical comparison of the performance of the heuristic policy to the optimal and to the compliance table policy. We observed that for the majority of the studied penalty functions, the heuristic policy outperformed the compliance table policy on most performance indicators.