Chapter 4

The need for speed: 
all-in-one versus iterative-shortcut approaches for solving general equilibrium 
models of urban transportation and land use

4.1. Introduction

General Equilibrium models of Land-Use and Transport (to which we will hereafter refer to as GELUT), as conceived and developed in the contributions of Anas and Kim (1996), Anas and Xu, (1997), Anas and Liu (2007) and applied empirically by Anas and Hiramatsu (2012; 2013) are becoming established as a widely accepted framework to model the urban economic environment. The approach provides a consistent quantitative method to perform welfare analyses of transport investments, land-use regulations, externality-pricing policies and reforms of a local tax system.

Such urban general equilibrium models have become recognized as alternatives to spatial interaction/gravity models (Mackett, 1991), Land Use and Transport Interaction (LUTI) models (see for instance TIGRIS-XL by Zondag and de Jong, 2011 to mention only one) and disequilibrium (i.e. adjustment-based) micro-simulation approaches such as UrbanSim (Waddell, 1998c, 2000, 2002; Waddell et al., 2003). The latter approaches deviate from a general equilibrium approach in many ways. Some of them treat production and household income as exogenous. Others do not depend on endogenously derived demand and supply functions and may rely instead on ad hoc price setting mechanisms. Equipped with these mechanisms, LUTI models are intrinsically capable of replicating the observed correlation between many important variables of the urban economy (e.g. residential and employment density). For this reason, they are (in general) characterized by larger forecasting power compared to fully microfounded approaches. However, the eclectic sampling of behavioral mechanisms from the systematic and logical panoply of microeconomic theory translates into a reduced capacity to perform reliable welfare analysis. Because of this state-of-affairs, a wide gap (that includes too many aspects to cover meaningfully in one chapter) exists between GELUT models and the above approaches.

Many years after the conception of the GELUT approach, its true possibilities and limitations have not yet been fully explored. Knowledge over their properties and mechanics accumulates through sporadic applications, or from the closely related field of Computable

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73 This chapter is a major extension of an earlier version of Anas and Tikoudis (2015) that opened the Applied Urban Modeling conference, 24-26 June 2015, which was held in University of Cambridge, UK. I am grateful to the local organizers, especially Dr. Ying Jin, for financial support.

74 See Iacono et al. (2008) for a comprehensive taxonomy (in both operational as well as historical terms) of models for transportation and land-use change.
General Equilibrium (hereafter, CGE). One of the most important challenges GELUT models face regards their solution. They usually require a high degree of numerical precision in order to obtain consistent predictions and provide accurate welfare analyses. Furthermore, many research/policy questions pertain to the evaluation of small-scale policy interventions, such as the creation of small open spaces in multiple locations of a city. Other research questions regard spatially-detailed welfare analyses of arbitrary policy interventions. Consider for instance the construction of a ring-road and the evaluation of noise and pollution externalities that fade out a few hundred meters away from it.

Elaborate answers to such questions require detailed GELUT models of high spatial resolution, something that increases the computational burden faced by the modeler, rendering in many cases computational speed a major issue. An unfortunate result is that GELUT models are then considered too difficult or time demanding to solve and hence impractical to use. This chapter focuses entirely on the construction of a representative all-in-one GELUT model, in which the choices of household location, employment location, route and mode are simultaneous. It guides the reader through the economic specification of the model and offers two different numerical solution techniques, which should be reasonably easy to replicate.

The first technique, to which we refer as the all-in-one approach, is an accurate, but computationally cumbersome approach that solves simultaneously for the economic and the stochastic user equilibrium in transport network. In this equilibrium all markets of the model (i.e. final and intermediate products, housing of all types and land) clear and, due to free entry and perfect competition, profits in all commodities (apart from the primary factors) are zero. The economic equilibrium contemplates the land-use equilibrium because land, housing prices and building stocks are codetermined with the rest of the prices and quantities in the urban system. The stochastic user equilibrium ensures that mode and route choice probabilities are consistent with the traffic conditions (travel times and costs) they give rise to.

The second technique is speed-driven and attempts to approximate the all-in-one model with a different behavioral model (but still GELUT), in which the location choices are detached (taken at different time points) from the mode and route choices. We refer to this approximation model as the iterative-shortcut model. The corresponding algorithm iterates between the two equilibria (hence iterative). Because traffic conditions (i.e. travel times and costs) and route choice probabilities are fixed during the solution of the economic model, weighted travel times and costs can be computed for each origin destination pair (OD pair). This gives the possibility to reconstruct and solve the household optimization problem at the OD pair level, rather than the route-and-mode level (as is the case in the all-in-one approach). This drastic reduction in the size of the household’s choice set provides a major shortcut in terms of computational time. The shortcut becomes larger as the total number of routes (including multimodal routes) relative to the number of OD pairs, hereafter the link-based resolution of the model, increases (i.e. as new links are added in a fixed set of OD pairs).

We provide the details behind the algorithms of the two procedures, elaborate schematic depictions (flow charts), as well as pseudo code disengaged from any programming language.
We focus on the computational efficiency of the *iterative-shortcut* approach relative to the *all-in-one* simultaneous solution. We show that the relative computational time (*i.e.* the ratio between the time required for the *all-in-one* solution and the respective time for the *iterative-shortcut* solution) is increasing in the link-based resolution. However, the equilibrium reached in the solution of the *iterative-shortcut* approximation model is, in general, not identical to the equilibrium reached in the solution of the *all-in-one* model, as the two specifications do not point to the same household optimization problem. This is because the aggregate demand functions produced by the two behavioral models differ from each other. We, however, highlight the special conditions under which the *iterative-shortcut* approximation model can be reformulated as an *all-in-one*. We prove that under these conditions the two approaches reach the same equilibrium.

The structure of the chapter is as follows: Section 4.2 provides an elaborate example of a general *all-in-one* GELUT model. Section 4.3.1 discusses its solution with the *all-in-one* algorithm and section 4.3.2 the approximation of the *all-in-one* model with the *iterative-shortcut* model, as well as the solution of the latter model with the *iterative-shortcut* algorithm. Section 4.3.3 provides computational examples with four different network configurations. Section 4.4 summarizes and concludes.

4.2. Constructing a sample all-in-one GELUT model

This section introduces the generic setting and the main actors of a GELUT model (*i.e.* utility-maximizing households, profit-maximizing firms and a benevolent government). It describes the basic behavior that is common in such models, as well as the underlying mechanisms that support a static equilibrium.

4.2.1. Space, network representation and discrete choice

Economic activity takes place in an ordered set of \( J \) zones (each represented by a node of a transport network), \( J \). Let the ordered subsets \( J_R \) and \( J_W \) denote the locations that host residences and jobs respectively, with \( J = J_R \cup J_W \). Throughout the text, the subscript \( i \) is used to denote an arbitrary zone in the ordered set \( J \) that serves as a residential node, *i.e.* \( i \in J_R \). Similarly, the subscript \( j \) is used to denote an arbitrary zone in the ordered set \( J \) that serves as an employment node, *i.e.* \( j \in J_W \). Mixed land-use may be possible, in this case \( J_R \cap J_W \neq \emptyset \). Let the set \( C_{OD} = J_R \times J_W \) denote the Cartesian product of sets \( J_R \) and \( J_W \), *i.e.* the set that contains all possible pairs of residential and employment locations. Each element \( c_{ij} \in C_{OD} \) is an origin-destination pair (hereafter, we will refer to such a zone pair as *OD pair*).

Two arbitrary zones, \( s \) and \( e \), are neighboring if there is at least one transport link \( l_{m}^{(se)} \) starting at \( s \) and ending at \( e \), where the subscript \( m \) denotes the type of transport network the link belongs to (*e.g.* road, rail, *etc.*). Links are directed, thus \( l_{m}^{(se)} \neq l_{m}^{(es)} \). A route \( q \) is defined as a
sequence (ordered list) of links such that, for each pair of consecutive links in the sequence, \( l_m^{(se)} \) and \( l_m^{(s'e')} \), it holds that \( e = s' \), although it can be that \( m \neq m' \) (for multimodal paths).

For each OD pair \( a_{ij} \) in \( \mathcal{C}_{OD} \) there is a set of corresponding possible routes that connect the origin with the destination zone, which we denote by \( Q(a_{ij}) \). It is straightforward that, if origin zone, \( i \), and destination zone, \( j \), are neighboring, then it holds that any \( l_m^{(ij)} \in Q(a_{ij}) \). An alternative, \( a \), is a set that contains the OD pair \( a_{ij} \) and a route \( q \in Q(a_{ij}) \); we denote it by: \( a = \{a_{ij}, q\} = \{i, j, q\} \). The choice set, denoted by \( \mathcal{C} \), contains all possible alternatives (i.e. zone pairs and route combinations).

4.2.2. Household behavior

GELUT models are discrete-continuous in their nature. The discrete part involves a choice of an alternative \( a = \{a_{ij}, q\} \) from the choice set \( \mathcal{C} \). Conditional on this choice, the household chooses a series of continuous endogenous variables subject to budget and time constraints which depend on the choice of alternative \( a = \{a_{ij}, q\} \). The latter determines, as it becomes clear below, the relative prices faced by household. What we present below may be regarded as a simplified version of Anas and Xu (1999). But unlike that model, in this setting we assume that all traffic is generated by households that commute every working day. Non-work trips (e.g. shopping) are assumed away by supposing the existence of a homogeneous good that is made available to consumers at zero transport cost. As in Anas and Xu (1999), we assume a single income group of households with idiosyncratic taste heterogeneity within the group.

In this example, each household has a single working member and, therefore, only one commuter. Time is allocated between commuting, leisure and labor supply which is adjustable through the number of working days, \( D_{Wa} \), but not through the duration of a given working day; the latter is assumed to be fixed and we normalize it to one. Therefore, labor supply adjusts exclusively at the extensive margin, as in Verhoef (2005), Tscharaktschiew and Hirte (2010) and Tikoudis et al. (2015b). Each day at work implies one commute; denoting commuting time by \( t_a \), the total time spent on labor and commuting under the choice of alternative \( a \) is: \( D_{Wa}(1 + t_a) \).

All leisure activities are homogenous and therefore there is a unique leisure time, \( T_{Fa} \). The time constraint takes the form:

\[
T = D_{Wa}(1 + t_a) + T_{Fa}.
\]  \hspace{1cm} (4.1)

We assume a unique residential type.\textsuperscript{75} Net labor income per unit of labor supply is the daily disposable wage (i.e. \( w_j \) multiplied by one minus the income tax rate \( \tau_L \)) minus the daily pecuniary commuting cost, \( c_a \), and toll, \( \tau_{Ra} \) of route \( q \) embodied in the alternative \( a = \{a_{ij}, q\} \). If

\textsuperscript{75} More elaborate models can be constructed to describe household behavior, for instance Anas and Liu (2007) and Tscharaktschiew and Hirte (2010). These may include choice among multiple residential types (single-family, multi-family residences), the choice of car ownership.
the chosen alternative involves an OD pair \( a_{ij} \) such that \( i = j \), then no commuting occurs, the list \( q \) is empty and \( c_a, t_a \) and \( \tau_{Ra} \) are all set equal to zero.

The household's budget constraint is:

\[
y_a + p_{Hi} s_a = e + D_{Wa} [w_j (1 - \tau_L) - c_a - \tau_{Ra}],
\]

(4.2)

where \( y_a \) denotes the quantity of a numéraire composite commodity, \( p_{Hi} \) the price per unit of floor space, \( s_a \) the housing floor space, and \( e \) the exogenous to the household, non-labor income. Note that, because (in this context) each route \( q \) is part of a unique alternative, \( c_a, t_a \) and \( \tau_{Ra} \) can be used interchangeably with \( c_q, t_q \) and \( \tau_{Rq} \). Solving (4.1) for \( D_{Wa} \) and inserting it to (4.2) yields the full time constraint:

\[
y_a + p_{Hi} s_a + v_a T_Fa = e + v_a T,
\]

(4.3)

where

\[
v_a = \frac{w_j (1 - \tau_L) - c_a - \tau_{Ra}}{1 + t_a},
\]

(4.4)

is the shadow value of time and \( v_a T \) is the household's full income conditional on the choice of \( a = \{a_{ij}, q\} \). Note that the value of time is homogenous for a given alternative \( a = \{a_{ij}, q\} \). This is an artifact of the adopted assumptions: more complicated specifications can result in values of leisure time deviating from values of working time, with the latter depending on the duration of the working day, and the former on whether leisure succeeds a working day or fills up an entire day out of work.

However, the great advantage of GELUT models becomes apparent even in simplistic specifications. In contrast to other approaches, GELUT models produce endogenous time valuations that are functions of characteristics in related markets, traffic conditions and related policy instruments: \( v_a \) varies across alternatives because the gross wage, commuting time, pecuniary costs and road tolls vary across the choice set faced by the household.

We switch-off income effects by specifying a quasi-linear utility function:

\[
U_a = \pi_0 y_a + \pi_1 \left( s_a^\alpha \tau_Fa^\beta \right)^\gamma,
\]

(4.5)

with the marginal utility of income being constant and equal to \( \pi_0 \).\(^76\) We assume that \( \alpha + \beta = 1 \) and \( \gamma < 1 \). Maximizing (4.5) subject to (4.3) yields the Marshallian demand functions for housing and leisure time respectively, for any alternative \( a \):

\(^76\) Later on, we discuss the rationale behind the choice of a preference relation characterized by constant marginal utility of income.
Inserting (4.7) into (4.1) yields the optimal labor supply for alternative a:

\[ D_{Wa}^* = \frac{T - \left( \frac{p_H \pi_0}{\alpha \gamma \pi_1} \right)^{\frac{1}{\gamma - 1}} \left( \frac{\alpha v_a}{\beta p_H} \right)^{\frac{\beta}{\gamma - 1}}} {1 + t_a}. \]  

Optimal consumption, \( y_a^* \), can be computed by inserting (4.6) and (4.7) into (4.3). In Appendix 4.F it is shown that this is:

\[ y_a^* = e + v_a T - \frac{\alpha v_a}{p_H \gamma - 1} \left( \frac{\pi_0}{\alpha \gamma \pi_1} \right)^{\frac{1}{\gamma - 1}} \left( \frac{\alpha}{\beta} \right)^{\frac{\beta}{\gamma - 1}}. \]  

Substituting \( y_a^* \), \( s_a^* \) and \( T_{Fa}^* \) into the objective function yields the indirect utility of alternative a which can be shown (Appendix 4.F) to be equal to: \(^{77}\)

\[ V_a^* = \pi_0 (e + v_a T) + \left( \left( \frac{\alpha}{\beta} \right)^{\frac{\beta}{\gamma - 1}} \right) + \left( \frac{\alpha}{p_H \gamma - 1} \right) \Gamma, \]  

where

\[ \Gamma = \pi_0^{\frac{\gamma}{\gamma - 1}} \pi_1^{\frac{1}{\gamma - 1}} \left\{ \frac{\gamma}{\gamma - 1} \left( \frac{\alpha}{\beta} \right)^{\frac{\beta}{\gamma - 1}} \left( \frac{p_H}{\gamma - 1} \right)^{\frac{1}{\gamma - 1}} \right\}. \]

is the parametric function of \( \alpha, \beta, \gamma, \pi_0, \) and \( \pi_1 \). Now, inserting (4.4) into (4.10) yields:

\[ V_a^* = z_a + \pi_0 \left( e + \frac{w_f (1 - \tau_L) - c_a - \tau_R a}{1 + t_a} T \right), \]

\[ + \left( \frac{w_f (1 - \tau_L) - c_a - \tau_R a}{1 + t_a} \right)^{\frac{\beta}{\gamma - 1}} p_H^{\frac{\alpha}{\gamma - 1}} \Gamma. \]

\(^{77}\) See Chapter 5 for further discussion over the use of the alternative-specific constant. The closed form expression of the indirect utility of alternative a and its cumbersome derivation are provided in Appendix 4.F.
Finally, a stochastic term, \( \epsilon_a \), which is *i.i.d. extreme value type I* across alternatives in \( \mathcal{C} \), is added to (4.12) in order to capture the rest of the factors that are omitted in the model and may determine the choice of \( a = \{a_{ij}, q\} \). Total utility is, thus:

\[
U_a^* = \pi_0(e + v_a^T) + \frac{by}{v_a^T} p_{HI} \frac{av}{v^T} + \epsilon_a. \tag{4.13}
\]

Because the error component \( \epsilon_a \) is additive to the systematic utility and follows a *Generalized Extreme Value* distribution, the expectation of the maximum utility (hereafter, \( E_{\text{max}} \)) that can be derived when facing the choice set \( \mathcal{C} \) is the well-known as *logsum*, closed form expression:

\[
E_{\text{max}} = \lambda \left[ \mathcal{E} + \log \sum_{a \in \mathcal{C}} \left( \frac{\exp(V_a^*)}{\lambda} \right) \right], \tag{4.14}
\]

where \( \mathcal{E} \approx 0.5772 \) is the Euler constant. Under the assumption that \( \epsilon_a \) is *iid EV1* across alternatives \( a \in \mathcal{C}, i.e. \) across all residential and working locations, as well as commuting routes, the choice probability of alternative \( a \) is:

\[
\pi_a = \frac{\exp(V_a^*)/\lambda}{\sum_{\xi \in \mathcal{C}} \left( \exp(V_\xi^*)/\lambda \right)}, \tag{4.15}
\]

where the scale parameter \( \lambda \) is equal to \( \sqrt{\text{Var}(\epsilon_a)} \), *i.e.* the standard deviation of the distribution of random utility terms.

### 4.2.3. Firms, developers and assembly industry

On the production side, we assume that a unique intermediate good is produced in each zone of the model where employment/production is allowed. Intermediate goods are produced using the labor supplied by the households and imported capital. An *assembly* (final goods) industry then combines all intermediate goods to produce the final good which is ubiquitously available to all consumers at zero transport cost. We use the word *assembly* to describe an abstract industry that is not located anywhere and does not add value to the existing intermediates. That is, the final good is produced without the use of human labor, capital or land, but only intermediates. The final good is the composite good consumed by the households but it is also exported to the *rest of the world* (hereafter, \( \text{ROW} \)) in exchange for (i) capital and (ii) for another good that pays for the pecuniary costs of commuting. Real estate developers, are assumed to produce housing floor space in each residential zone using land and imported capital as inputs.\(^{78}\)

\(^{78}\) For practical reasons the representation of production is simplified considerably compared to that used in Anas and Liu (2007). In that model, each zone produces a differentiated final good using labor, capital and floor space as
A profit-maximizing, representative firm is located in each location $j \in \mathcal{J}_W$ and produces a homogenous output, $Q_j$, under constant returns to scale. The production function in its general form is:

$$Q_j^S = A_j \cdot f_F(K_j^{D_f}, L_j^D),$$

(4.16)

in which $K_j^{D_f}$ and $L_j^D$ denote the capital input and labor supplied to zone $j$ by the households residing in all residential zones respectively; $A_j$ is the location-specific total factor productivity.\(^{79}\) This term encapsulates the geographic, idiosyncratic advantages or disadvantages of a given zone.\(^{80}\) For any arbitrary level of output, $Q_j^S$, the firm is minimizing the cost: $RK_j^{D_f} + w_jL_j^D$ subject to the production function in (4.16). This yields the cost function:

$$c_j^F(Q_j^S) = c_j^F(R, w_j, A_j | Q_j^S).$$

(4.17)

where $R$ denotes the price of capital and $w_j$ the wage in zone $j$.\(^{81}\) The conditional factor demands for capital and labor can be derived using Shephard’s lemma:

$$K_j^{D_f}(R, w_j, A_j | Q_j^S) = \frac{\partial c_j^F(R, w_j, A_j | Q_j^S)}{\partial R},$$

(4.18)

$$L_j^D(R, w_j, A_j | Q_j^S) = \frac{\partial c_j^F(R, w_j, A_j | Q_j^S)}{\partial w_j},$$

(4.19)

Perfect competition draws profits to zero, therefore the market price of each locally-produced good, $p_{lj}$, equals the marginal cost of the good:

$$p_{lj} = \frac{\partial c_j^F(R, w_j, A_j | Q_j^S)}{\partial Q_j^S}. $$

(4.20)

\(^{79}\) The production function’s ($f_F$) subscript is used to differentiate it from the production functions used to describe the construction sector, $f_D$ (see below).

\(^{80}\) In a more elaborate version of the model, total factor productivity may be specified to arise as a product of job agglomeration in the region. That is, agglomeration externalities can be further modeled explicitly by specifying $A_j$ as a function of job density in zone $j$.

\(^{81}\) If we further assume constant returns to scale, (4.17) can be written as: $c_j^F = \frac{Q_j}{A_j} c(R, w_j)$.
An assembly industry combines the intermediates produced by local firms into a final, regional good: the numéraire (or the composite good), whose price we normalize to one. The production function is:

\[ Y^S = f_A(Q^A_j), \tag{4.21} \]

and the respective cost function is:

\[ c^A = c^A(p_I|Y^S). \tag{4.22} \]

The conditional demand for the intermediate good produced in zone \( j \) from the assembly industry is:

\[ Q^A_j(p_I|Y^S) = \frac{\partial c^A(p_I|Y^S)}{\partial p_{ij}}. \tag{4.23} \]

Finally, the zero profit condition is:

\[ p = \frac{\partial c^A(p_I|Y^S)}{\partial Y^S}. \tag{4.24} \]

The construction sector is also assumed to be competitive. We assume the existence of a representative developer in each zone \( i \in J_R \), whose behavior is identical to that of profit maximizing firms. Each developer converts land and capital into a unique type of floor space (i.e. a unique residential type). The production function for floor space of this type is:

\[ h^S_i = f_D(K^D_i, X^D_i), \tag{4.25} \]

where the superscript \( D_d \) is now used to denote that the quantities are demanded by a developer (in contrast to \( D_f \) of the previous section that was used to denote demand by firms). Minimizing the cost \( RK^D_i + p_{Li}X^D_i \) subject to (4.25) yields the cost function:

\[ c^D(h^S_i) = c^D(R, p_{Li}|h^S_i). \tag{4.26} \]

Again, Shephard's lemma yields the conditional factor demands for capital and land for each building type in:

\[ K^D_i(R, p_{Li}|h^S_i) = \frac{\partial c^D(R, p_{Li}|h^S_i)}{\partial R}, \tag{4.27} \]

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\[ ^{82} \text{We have abstracted from including human labor in the production functions of the construction sector. If combined with a unique type of labor, such an inclusion would require wages (and therefore the marginal products of labor) across representative firms and developers in any zone } j \text{ to be equated.} \]
\[ X^D_i(R, p_{li}|h_i^S) = \frac{\partial c^D(R, p_{lj}|h_j^S)}{\partial p_{li}}. \]  

(4.28)

The zero profit condition in zone \( i \) is:

\[ p_{hi} = \frac{\partial c^D(R, p_{li}|h_i^S)}{\partial h_i^S}. \]  

(4.29)

### 4.2.4. Transport system

Labor supply (\( D_{Wa} \)) is key because it determines completely link flows. Demand for an arbitrary link, \( l_m^{(se)} \), is given by:

\[ D_m^{(se)} = \sum_{a \in C} [\pi_a \times D_{Wa} \times I(q, l_m^{(se)})], \]  

(4.30)

where \( I(q, l_m^{(se)}) \) denotes the indicator function that takes the value one if the choice of alternative \( a = \{a_{ij}, q\} \) implies (through the choice of route \( q \)) the use of link \( l_m^{(se)} \) (zero otherwise). Assuming a linear volume-delay function for links in the road network (subscript \( m \) replaced by \( C \)):

\[ t_C^{(se)} = \ell_C^{(se)} (\tau_{0C}^{se} + \tau_{1C}^{se} D_C^{(se)}), \]  

(4.31)

where the intercept parameter, \( \tau_{0C}^{se} \), and the length of the road link, \( \ell_C^{se} \), determine the free-flow travel time via \( \tau_{0C}^{se} \ell_C^{se} \), and \( \tau_{1C}^{se} \) denotes the sensitivity (slope) of link travel time to increments in its flow, \( D_C^{(se)} \). For simplicity, we assume that public transport modes (subscript \( m \) replaced by \( R \)) are not subject to congestion, thus:

\[ t_R^{(se)} = \ell_R^{se} \tau_{0R}^{se}. \]  

(4.32)

Adding the travel time of the link components of an arbitrary route, \( q \), yields its travel time. This is:

\[ \hat{t}_q = \sum_{l_m^{(se)} \in q} t_m^{(se)}. \]  

(4.33)

The total pecuniary cost of a route is:

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83 The choice of a linear volume-delay function is arbitrarily made for expositional reasons: any other function could be chosen without affecting qualitatively the results of this chapter.
\begin{equation}
\begin{aligned}
c_q = p_g & \left\{ \sum_{l_C^{(se)} \in q} (r_C^{(se)}) \right\} + p_P \left\{ \sum_{l_R^{(se)} \in q} (r_R^{(se)}) \right\},
\end{aligned}
\end{equation}

where \( p_g \) and \( p_P \) are the exogenous prices (\textit{i.e.} determined in the rest of the world) per unit of distance travelled with car and public transport respectively.

4.2.5. Government and public budget

We assume that the government may impose road pricing schemes (discussed below) and recycles the tax and road toll revenues. The total labor tax paid by the households is equal to:

\begin{equation}
G_L = N \tau_L \sum_{a \in C} (\pi_a D_W a w_j).
\end{equation}

Similarly, total toll revenue is:

\begin{equation}
G_R = N \sum_{a \in C} (\pi_a D_W a \tau_{Ra}),
\end{equation}

where

\begin{equation}
\tau_{Ra} = \tau_{Rq} = \sum_{l_C^{(se)} \in q} \tau(l_C^{(se)}),
\end{equation}

and \( \tau(l_C^{(se)}) \) denotes the road toll imposed by government on road link \( l_C^{(se)} \). Road pricing policies consist of two components: (i) an assignment of a unique road charge to each road link \( l_C^{(se)} \), and (ii) a description of how total tax revenue, \textit{i.e.} \( G_R \) is returned to society. From this point, we are referring to these policy components as \textit{road toll scheme} and \textit{revenue recycling} respectively. The government runs a balanced budget, therefore:

\begin{equation}
\frac{1}{N} (G_R + G_L) = B,
\end{equation}

where \( B \) is a \textit{tax-financed lump-sum transfer} from the government to the household (see below). Moreover, the government owns the land which is conceded to developers. The aggregate land

\[84\] Alternatively, we could assume a government obtaining the land from an absentee landlord and then selling it at its equilibrium market price to developers, while recycling the profits (\textit{i.e.} the aggregate differential land rents) to the households of the region. Such an assumption would not affect any qualitative property of the model.
rents are also returned to the \( N \) households of the region. The *land dividend* that corresponds to each household is:

\[
\frac{1}{N} \sum_{i \in J_R} (p_{Li} \bar{X}_i) = B_{\ell}, \tag{4.39}
\]

where \( \bar{X}_i \) is the total surface of zone \( i \) and \( B_{\ell} \) is the land income of each household.\(^{85}\) The exogenous income of household (see Section 4.2.2) is the sum of the tax-financed lump-sum transfer and the land dividend, thus:

\[
e = B + B_{\ell}. \tag{4.40}
\]

### 4.2.6. General, stochastic user equilibrium

The competitive equilibrium requires four sets of conditions. *First*, all expectations regarding \( c_a \), \( t_a \) and \( \tau_{Ra} \) must be met. This means that the values of these variables used to compute (4.6)-(4.12), give rise to choice probabilities and aggregate variables that, when used in (4.30)-(4.33), (4.15) and (4.37), reproduce exactly these times and costs. This implies that:

\[
\hat{t}_q - t_q = 0, \tag{4.41}
\]

*Second*, all local firms make zero profits. This implies \( f_W \) zero profit conditions, one for each representative firm located in an employment zone. There are also \( J_R \) zero profit conditions as those in (4.29) for the developers. Finally, the assembly industry is also competitive, thus condition (4.24) has to be added in the list of requirements for a competitive equilibrium.

The *third* set of equations consists of clearing equations for the markets of land, floor space, labor and intermediates. Particularly for the housing market for each zone \( i \), clearing implies that:

\[
h_{i}^S = N \sum_{a \in c} (\pi_a s_a) \tag{4.42}
\]

where \( h_{i}^S \) denotes residential space supply at zone \( i \). Since we have assumed a unique residential type, here there are \( J_R \) such clearing conditions.\(^{86}\) The market clearing condition for land in each zone in \( J_R \) is:

\[^{85}\text{Alternatively, we could have specified only a proportion of the land surface in each zone to be owned by the households, as in Anas (2007).}\]

\[^{86}\text{More complex models can assume multiple residential types, with only some of them or one of them being allowed to develop in a certain zone. The number of residential floor space clearing equations is therefore determined by the assumptions made in a PGE-tlu model.}\]
\[ \bar{X}_i = X^D_i (R, p_{Ll} | h_i^S). \] (4.43)

There are \( J_R \) such clearing conditions. The clearing condition for the labor market in each zone in \( J_W \) is:

\[ N \sum_{a \in C} \pi_a \times I(i, j, a) \times D_{Wa} = L^D_j (R, w_j, A_j | Q_j^S), \] (4.44)

Also, the market for each intermediate \( Q_j \) (produced by the local firm in zone \( j \) and sold to the assembly industry) must clear. The associated condition here is:

\[ Q_j^S = Q_j^{D_A} (p_I | Y^S), \] (4.45)

which implies another \( J_W \) clearing conditions. We conclude with the fourth type of condition, as we are now ready to close the model.

### 4.2.7 Model closure

We have already noted that the urban economy is open to the market for capital, used by developers and intermediate good producers. Moreover, households must obtain (through imports) the inputs from which pecuniary costs of transportation are defrayed (including gasoline, vehicle depreciation, public transport provision etc.). The total value of inputs from the rest of the world generated in zone \( k \in J \) is:

\[ \Psi_k = R \left[ I(k \in J_W) \cdot K^{D_f}_i (R, w_j, A_j | Q_j^S) + I(k \in J_R) \cdot K^{D_d}_i (R, p_{Ll} | h_i^S) \right] \]

\[ + N \sum_{a \in C} \pi_a \times I(i = k, a) \times D_{Wa} \times c_q, \] (4.46)

where the indicator functions \( I(k \in J_W) \) and \( I(k \in J_R) \) equal one if zone \( k \) belongs to sets \( J_W \) and \( J_R \) respectively (zero otherwise), and the indicator function \( I(i = k, a) \) equals one if the alternative \( a = \{ a_{ij}, q \} \) implies zone \( k \) as origin, i.e. if \( a = \{ a_{kj}, q \} \) (zero otherwise). The final good produced by the urban economy is traded in the world market to obtain the above inputs.

Therefore, a balanced-trade condition must be incorporated in the model. The closure condition requires the total value of imports in the model, i.e. \( \sum_{k \in J} \Psi_k \), to be balanced by an equal value of exports to the rest of the world. Therefore we impose:
\[ p(Y^S - Y^D) = \sum_{k \in J} \Psi_k, \quad (4.47) \]

where \( p \) denotes the price of the good which functions as a numéraire (which we discuss below) and \( Y^D \) is the total demand for the final, composite good produced by the regional assembly industry:

\[ Y^D = N \sum_{a \in C} (\pi_a y_a). \quad (4.48) \]

Condition (4.47) replaces the typical clearing condition for the numéraire:

\[ Y^S - Y^D = 0, \quad (4.49) \]

that could be correctly imposed in a completely closed economy.\(^{87}\)

### 4.2.8. Equation count

The specification of the model, as presented in Sections 4.2.1-4.2.7, yields a square system of non-linear equations to be solved. For each alternative \( a \in \mathcal{C} \), we have: the Marshallian demand for space and leisure time in equations (4.6) and (4.7), the Marshallian labor supply in (4.8), the Marshallian demand for the composite final good (as inferred from (4.2)), the systematic (indirect) utility in (4.12) and the choice probability in (4.15). Furthermore, for each employment zone \( j \in \mathcal{J}_W \) we have: the conditional factor demands (for capital in (4.18) and labor in (4.19)) and the zero profit condition for the representative firm in (4.20). For each residential zone \( i \in \mathcal{J}_R \) we have: the conditional factor demands (for capital in (4.27) and land in (4.28)) and the zero profit condition in (4.29). For the assembly industry we have: one conditional factor demand for each intermediate (\( i.e. \) for each employment zone \( j \in \mathcal{J}_W \)) in (4.23) and the zero profit condition in (4.24). The market clearing conditions include \( J_R \) equations as in (4.42) for the housing market, \( J_R \) equations as in (4.43) for the land market, \( J_W \) equations as in (4.44) for the labor market and \( J_W \) equations as in (4.45) for the intermediate goods.

For the transport system we have: one flow equation as in (4.30) for each link, a volume delay function as in (4.31) for each road link, a travel time equation as in (4.32) for each public

---

\(^{87}\) Specifying the closure condition as a market clearing condition (by assuming that the import costs have been incurred directly by firms, developers and households) is a typical pitfall in such a model. It occurs because the model fails to account explicitly for the agent that imports the goods produced in \( ROW \) and consumed inside the model. Assuming free entry in the import sector, the representative importer will make zero profits by borrowing part of the final output produced in the model, \( Q \), which he repays with imported goods of equal value. These imported goods belong now to the industries that produced the foregone output, which sell them to those that demand them. Therefore in any equilibrium that supports imports, the value of final production and consumption cannot be the same.
transport link, commuting times for each alternative in (4.33), the total road tax per route (one equation like in (4.37) for each commuting alternative), the pecuniary cost per route (one equation like in (4.34) for each commuting alternative), as well as the route time consistency equations (one equation as in (4.41) for each route).

Finally, we have the labor tax revenue in (4.35), the road tax revenue in (4.36), the tax revenue-recycling equation in (4.38), the land rent redistribution equation in (4.39), the exogenous income equation in (4.40), the aggregated demand for final goods in equation (4.48), the total value of imports (one equation as in (4.46) for each zone of the model), the expected maximum utility in (4.14) and the macroclosure condition, given by equation (4.47).

In total, the equilibrium is characterized by a solution of 34 blocks of equations. There are 34 corresponding vectors of endogenous variables. For each alternative \( a \in \mathcal{C} \), we have: \( y_a \), \( s_a \), \( T_{Fa} \), \( D_{Wa} \), \( V_a^* \), \( \pi_a \); for each employment zone \( j \in J_W \) we have: \( Q_j^S \), \( K_j^D \), \( L_j^D \), \( p_{ij} \), \( Q_j^{DA} \), \( w_j \); for each residential zone \( i \in J_R \): \( h_i^S \), \( K_i^{Da} \), \( X_i^{Da} \), \( p_{Li} \), \( p_{Hi} \); for each link (of any type): \( D_m^{(se)} \); for each road link: \( t_c^{(se)} \); for each public transport link: \( t_R^{(se)} \); for each route: \( \hat{t}_a \), \( t_q \), \( c_q \), \( \tau_{Rq} \); and finally the variables: \( E_{\text{max}} \), \( p \), \( G_L \), \( G_R \), \( B \), \( B_i \), \( e \), \( Y^S \), \( Y^D \), \( \Psi_R \). Denoting by \( N_C \), \( N_F^e \), \( N_F^R \), and \( N_P \) the number of alternatives, road links, public transport links and routes respectively, the reader can confirm that the number of equations is: \( 6J_W + 5J_R + 7N_C + 2N_F^C + 2N_F^R + 3N_P + 10 \).

### 4.3. Solution approaches

We now discuss two different approaches in solving for the general, stochastic user equilibrium of the all-in-one model. The first approach is the exact solution of the all-in-one system described in Section 4.2. Because the system is highly non-linear, solution algorithms that attempt to solve all equations simultaneously are highly likely to fail, even with very good starting values. We propose a general approach that reduces the size of the system by encapsulating some of its equations into others. We refer to this method (algorithm) as all-in-one solution algorithm, because it solves for a simultaneous equilibrium both in land-use and transportation parts of the model. The all-in-one algorithm is discussed in Section 4.3.1.

The equations of the model increase rapidly as the resolution becomes higher. Adding new nodes implies larger choice sets, thus the continuous part of the household optimization problem has to be solved more times. This is because (i) the number of OD pairs and (ii) the number of routes available for a given OD pair both increase. Clearly, the computational burden is an obstacle to more elaborate, less stylized applications that can provide richer policy recommendations. The approximation of the all-in-one model by a different behavioral model, which is solved by an alternative algorithm, alleviates the computational burden significantly. In that approximation model, residential and employment location decisions are taken at different time points than decisions over the choice of mode and route. Then, the corresponding algorithm iterates between solving for an economic and a transport equilibrium separately. During the solution of the economic model, the expected (conditional on the OD-pair) travel times and costs
are kept fixed. This gives the possibility to reconstruct and solve the household optimization problem at the OD pair level, rather than the route-and-mode level (as it is the case in the all-in-one approach). This drastic reduction in the size of the household’s choice set provides a major shortcut in terms of computational time.

We refer to the above as the iterative-shortcut approach. Section 4.3.2 presents the approximation of the all-in-one model by an iterative-shortcut model and discusses the corresponding algorithm for the solution of the latter. We identify conditions under which the solution of the iterative-shortcut approximation model represents the same equilibrium as the one obtained by the all-in-one algorithm. We argue that even if this is not possible the iterative-shortcut approach can provide efficient starting values for the former. Section 4.3.3 examines the relative computational time in a series of network configurations.

4.3.1. The all-in-one solution algorithm

The highly non-linear system described above can, in general, be solved with the Newton method. However, this might require good starting values for all endogenous variables that may not be available. This is particularly true for utility values, $V^*_a$. Furthermore, it might be easier to work with smaller sub-systems. Consider first the square sub-system consisting of the blocks of equations (4.20), (4.24), (4.29), (4.40), (4.41), (4.42), (4.43), (4.44), (4.45), (4.47) (hereafter, we refer to these as the core equations) and the vectors of variables $p_{1j}$, $p_{Hi}$, $p$, $t_q$, $h^s_i$, $p_{Li}$, $w_j$, $Q^S_j$, $Y^S$ and $e$ (hereafter, we refer to the concatenation of these vectors as the core vector). We refer to this sub-system as the core sub-system.

A first observation is that knowing the variables of the core vector can completely determine the conditional factor demands in the blocks of equations (4.18), (4.19), (4.23), (4.27) and (4.28), the disaggregate labor supply, housing, leisure and composite demand for each alternative in the blocks of equations (4.2), (4.6), (4.7) and (4.8) as well as the redistributable land rents in equation (4.39). Furthermore, aggregate residential demand (the right hand side of (4.22)), aggregate labor supply (the left hand side of (4.44)), aggregate composite demand in equation (4.48) and the value of imports corresponding to each zone in equation (4.46) can be computed. But then the choice probabilities for each alternative can be computed using equation (4.15); subsequently, blocks of equations (4.30), (4.31), (4.32), (4.33) can be used to compute the actual travel time for each route, equations (4.35) and (4.36) to compute the revenue from labor and road taxation and equation (4.38) the tax recycling. We refer to this sequence of computations (described thoroughly in Appendix 4.B) as the inner process. After the inner process is completed, we can compute the core equations.

Therefore, an efficient algorithm that updates the core vector, then computes all the above equations (for which an analytic expression exists) and finally returns the values of the core equations, can be constructed. Appendix 4.B provides a schematic depiction of such an algorithm. The pseudocode is disengaged from the characteristics of any programming language,
no matter if this is procedural, object-oriented etc. However, it can be shown that as the resolution of the model increases (in the sense that more types of heterogeneous agents are included) an object-oriented design may be a safer practice for the modeler. The interested reader is referred to Tikoudis (2015) for an elaborate object-oriented design.

The demand and supply functions in the entire system of equations are homogenous of degree zero: aggregate demands, supplies and travel times will remain intact if all prices and wages are scaled by a constant. These prices include those that are exogenously determined because, although excess demand functions for the associated goods do not exist, Marshalian and conditional factor demands do. Therefore, the system has infinite solutions unless the price of one of these goods (i.e. the numéraire) is fixed through price normalization. As already seen, here we chose arbitrarily the composite good to function as a numéraire. Can capital, whose price in this setting is already exogenously fixed, act as a numéraire?

**Figure 4.1. Schematic depiction of the all-in-one algorithm.**

![Diagram](image)

The answer is that such a price normalization is not sufficient for closure. Although the excess demand functions under this normalization would not be homogenous of degree zero, we still haven’t made use of Walras’ law. That is, if excess demand functions for all but one good are zero, the market for the final good should clear as well. Therefore, we have to fix an arbitrary endogenous price and drop an arbitrary excess demand equation from the system to satisfy

---

88 All examples discussed in this chapter have been programmed in C# (Visual studio 2008). Code samples and further clarifications are available upon request.
Walras’ law. Subsequently, an imported good can act as a numéraire only if its price is determined endogenously. But in the specification considered in this chapter this is not the case as we have not modeled the rest of the world (ROW) explicitly.

The choice of numéraire good is arbitrary. In any case, the dropped equation functions as a specification test upon the completion of the algorithm displayed in Figure 4.1 and described in detail in Appendix 4.B. If the dropped equation does not hold but the algorithm has converged, this is a sign of a coding error or misspecification of the model (i.e. the behavioral equations might not reflect the objectives of firms and individuals).

4.3.2. The iterative-shortcut approach

4.3.2.1. The need for speed

Computational time needed for convergence depends, among others, on the number of zones and the type of network considered. Since solution time depends heavily on them, one may be interested in discovering faster algorithms than the one provided in Section 4.3.1 and Appendix 4.B. A quick look into the all-in-one algorithm reveals that the encapsulation (nesting of methods into each other) is such that the computation of Marshalian demands, indirect utility, choice probability, the traffic assignment and network loading are repeated every time an element of the core vector is altered.

Consequently, when new zones are added to the model, computational time is affected through three channels. First, the size of the core sub-system increases. Typically, each new zone augments the core vector \( \mathbf{o} \), adding market clearing and zero profit conditions in the existing core equations. Second, the number of core equations increases further because new routes are added in the model: every new route adds a new expectation-consistency condition in the core equations (i.e. that the difference between the input route travel time \( t_q \) and the resulting route travel time \( \hat{t}_q \), is zero in the stochastic user equilibrium). The number of new routes generated by an extra zone is usually larger in larger networks.

Most important, higher resolution increases the computational cost of the inner process, as the latter depends heavily on the size of the choice set. A larger choice set translates into more computations of equations (4.2), (4.6), (4.7), (4.8), (4.12), (4.15); furthermore, the denominator of (4.15), the computation of (4.35), (4.36), the right hand side of (4.42), (4.48), (4.46) and the left hand side of (4.44) become (computationally) more expensive.

---

89 Upon convergence, the remaining excess demand function can serve as a specification check: its value must be zero.
90 The number of new equations depends on whether the newly included zone is reserved for residences, jobs or both. There is at least one additional market to clear and at least one zero profit condition (with mixed land-use the number of additional equations is bigger).
4.3.2.2 A summary of the iterative-shortcut approach

The above problem motivates the construction of alternative approaches. The one presented here is iterative, in the sense that it focuses on the partition of the all-in-one equilibrium into an economic equilibrium and a stochastic user equilibrium that are reached iteratively, i.e. the one succeeds the other. This is achieved by approximating the all-in-one model by an alternative model in which the above detachment is possible. The economic equilibrium is reached without travel time expectations being realized, i.e. $\hat{t}_q \neq t_q$. Due to these conditions being excluded, the dimensions of the core syb-system reduce significantly, especially if the model contemplates a large number of zones and a mixed network.

Because travel times and costs are treated as exogenous during the solution of the economic equilibrium, the household optimization problem can be reconstructed at the OD pair level (see Section 4.3.2.3) rather than at the route level (as it is the case in the all-in-one model). This causes a drastic reduction in the size of the household’s choice set, providing a major shortcut in terms of computational time, speeding up the shortcut inner process (the corresponding version of the inner process discussed in Section 4.3.1). Upon convergence to the economic equilibrium, we solve for the stochastic user equilibrium that updates the OD-pair travel times and costs.

4.3.2.3 Approximating the all-in-one with an iterative-shortcut model: the special matching case

Suppose that the utility of the household’s full alternative $a = \{a_{ij}, q\}$, as expressed in equation (4.13), was additively separable in prices and wages, which are varying at the OD-pair level (subscripts $i$ and $j$), and in the travel times and costs, which are varying at the route level (subscript $q$). The systematic utility could then be decomposed as:

$$V_a^*(w_j, p_{Hi}, \tau_L, c_q, t_q, \tau_{Rq}, e) = V_a_{ij}^*(w_j, p_{Hi}, e, \tau_L) + \tilde{v}_q(c_q, t_q, \tau_{Rq}).$$

(4.50)

Under the above additive-separability assumption, the logit choice probability could then be rewritten as (Train, 2009):

$$\pi_a = \pi_{a_{ij}} \pi_q | a_{ij}.$$  

(4.51)

---

91 A further partition of the economic equilibrium according to groups of equations or endogenous variables (and a sequential solution of it with Gauss-Jacobi and Gauss-Seidel algorithm respectively) might also be possible. See for instance Anas and Liu (2007).

92 Here with mixed network we refer to one which is neither serial, nor parallel. For instance, consider a specification that includes 20 zones, 2500 feasible routes and allows for mixed land use in all zones (therefore 400 OD pairs). Then, the number of core equations (and variables) reduces from 2622 to 122.
where the marginal choice probability of the OD-pair \( a_{ij} \) is:

\[
\pi_{a_{ij}} = \frac{\exp(v_{a_{ij}}^* + \omega_{a_{ij}})/\lambda}{\sum_{a_{od} \in C_{OD}} \{\exp(v_{a_{od}}^* + \omega_{a_{od}})/\lambda\}}
\]  

(4.52)

the conditional (on \( a_{ij} \)) choice probability of route \( q \) is:

\[
\pi_{q|a_{ij}} = \frac{\exp(\tilde{v}_q/\lambda)}{\sum_{k \in Q(a_{ij})} \{\exp(\tilde{v}_k/\lambda)\}}
\]  

(4.53)

and the inclusive value of the OD pair \( a_{ij} \) is:

\[
\omega_{a_{ij}} = \lambda \left[ E + \log \sum_{k \in Q(a_{ij})} \{\exp(\tilde{v}_k/\lambda)\} \right].
\]  

(4.54)

Appendix 4.E shows that if (4.50) holds and the resulting Marshalian demands are separable in the two choice levels, then the iterative-shortcut algorithm will replicate the all-in-one. In this case, the two approaches mirror each other.

4.3.2.4. Approximating the all-in-one with an iterative-shortcut model: the general, non-matching case

We now show that even when the all-in-one approach does not have a mirror iterative-shortcut model (and vice versa), the latter can still be operationalized and (in large scale applications) provide tremendous computational gains. In this algorithm, we exploit the additional information obtained from the upper model (OD-pair choice) to approximate the unknown values of \( v_{a_{ij}}^* (w_j, p_{HI}, e, \tau_L) \) and \( \tilde{v}_q (c_q, t_q, \tau_R) \) in (4.50).

That is, we assume that conditional on the choice of an arbitrary OD pair \( (a_{ij}) \) households maximize the utility function:

\[
U_{a_{ij}} = \pi_0 y_{a_{ij}} + \pi_1 \left(s_{a_{ij}}^\alpha T_{Fa_{ij}}^\beta\right)^\gamma,
\]  

(4.55)

subject to the constraint:

\[
y_{a_{ij}} + p_{HI} s_{a_{ij}} + v_{a_{ij}} T_{Fa_{ij}} = e + v_{a_{ij}} T,
\]  

(4.56)

where:
\[ v_{a_{ij}} = \frac{w_j(1 - \tau_L) - c_{a_{ij}} - \tau_{Ra_{ij}}}{1 + t_{a_{ij}}} \]  

(4.57)

is the conditional (on the OD pair \(a_{ij}\)) value of time computed at the average commuting time \((t_{a_{ij}})\), cost \((c_{a_{ij}})\) and toll \((\tau_{Ra_{ij}})\) for this OD pair. The corresponding Marshallian demands for space and leisure time are:

\[
\begin{align*}
 s_{a_{ij}}^* &= \left( \frac{p_{Hi} \pi_0}{\alpha \gamma \pi_1} \right)^{1 \gamma - 1} \left( \frac{\alpha v_{a_{ij}}}{\beta p_{Hi}} \right)^{\beta \gamma - 1} \\
 T_{F a_{ij}}^* &= \left( \frac{p_{Hi} \pi_0}{\alpha \gamma \pi_1} \right)^{1 \gamma - 1} \left( \frac{\alpha v_{a_{ij}}}{\beta p_{Hi}} \right)^{\beta \gamma - 1} .
\end{align*}
\]

(4.58)

and

(4.59)

Labor supply is:

\[
D_{Wa_{ij}}^* = \frac{T - T_{F a_{ij}}^*}{1 + t_{a_{ij}}}. 
\]

(4.60)

Finally, \(y_{a_{ij}}^*\) can be computed from (4.56). Inserting the optimal values into the utility function yields the conditional indirect utility:

\[
V_{a_{ij}}^* = V \left( w_j, p_{Hi}, \tau_L, \tau_{Ra_{ij}}, t_{a_{ij}}, c_{a_{ij}}, e \right) = \\
= z_{a_{ij}} + \pi_0 \left( e + \frac{w_j(1 - \tau_L) - c_{a_{ij}} - \tau_{Ra_{ij}}}{1 + t_{a_{ij}}} T \right) + \left( \frac{w_j(1 - \tau_L) - c_{a_{ij}} - \tau_{Ra_{ij}}}{1 + t_{a_{ij}}} \right)^{\beta \gamma - 1} \frac{\alpha \gamma}{p_{Hi}^{\gamma - 1} \Gamma} .
\]

(4.61)

Now, we can approximate the unobserved OD pair portion of the indirect utility in (4.50), \(v_{a_{ij}}^*\), by the OD pair utility in (4.61). Then, we can approximate the unobserved route utility in (4.50), \(i.e. \bar{v}_q(c_q, t_q, \tau_{Rq})\), by the difference between \(V_{a_{ij}}^*\) and \(V_{a_{ij}}^*\):

\[
\xi_q = \frac{V_{a_{ij}}^* \left( w_j, p_{Hi}, \tau_L, c_{q'}, t_{q'}, \tau_{Rq'} \right) - V_{a_{ij}}^* \left( w_j, p_{Hi}, \tau_L, \tau_{Ra_{ij}}, t_{a_{ij}}, c_{a_{ij}}, e \right)}{\text{Total utility in (4.12)}} - \frac{V_{a_{ij}}^* \left( w_j, p_{Hi}, \tau_L, \tau_{Ra_{ij}}, t_{a_{ij}}, c_{a_{ij}}, e \right)}{\text{OD–pair utility in (4.61)}} .
\]

(4.62)

Replacing the unknowns \(v_{a_{ij}}^*\) and \(\bar{v}_q\) by \(V_{a_{ij}}^*\) and \(\xi_q\) in equations (4.52) to (4.54) yields the approximations \(\hat{\pi}_{a_{ij}}, \hat{\pi}_{q|a_{ij}}\) and \(\hat{\omega}_{a_{ij}}\) of \(\pi_{a_{ij}}, \pi_{q|a_{ij}}\) and \(\omega_{a_{ij}}\) respectively. These are:
\[
\begin{align*}
\hat{a}_{ij} & = \exp \left( V_{a_{od}}^* + \hat{a}_{od} \right) / \lambda \quad \Sigma_{a_{od} \in \mathcal{C}_{OD}} \left( \exp \left( V_{a_{od}}^* + \hat{a}_{od} \right) / \lambda \right) \quad (4.63) \\
\hat{q}_{a_{ij}} & = \frac{\exp (\xi_q / \lambda)}{\Sigma_k \in \mathcal{Q} \left( \exp (\xi_k / \lambda) \right)} \quad \Sigma_{k \in \mathcal{Q} \left( a_{ij} \right)} \{ \exp (\xi_k / \lambda) \} \quad (4.64) \\
\text{and:} \\
\hat{c}_{aij} & = \lambda \left[ \xi + \log \Sigma_{k \in \mathcal{Q} \left( a_{ij} \right)} \{ \exp (\xi_k / \lambda) \} \right]. \quad (4.65)
\end{align*}
\]

We now describe the core sub-system of the economic equilibrium. Condition (4.42) is replaced by:

\[
\begin{align*}
\hat{h}_{i}^S & = N \sum_{a_{ij} \in \mathcal{C}_{OD}} \left( \hat{a}_{ij} S_{a_{ij}} \right) \quad \text{(expected aggregate demand for residential space in zone i)} \\
\hat{a}_{od} & \times I(\cdot, j, a_{od}) \times D_{Wa_{od}} = \frac{L^D_{j}(R, w_j, A_j | Q_j^S)}{\text{(demand for labor in zone j)}} \quad (4.66)
\end{align*}
\]

Condition (4.44) is replaced by:

\[
\begin{align*}
N \Sigma_{a_{od} \in \mathcal{C}_{OD}} \hat{a}_{od} \times I(\cdot, j, a_{od}) \times D_{Wa_{od}} = \frac{L^D_{j}(R, w_j, A_j | Q_j^S)}{\text{(expected aggregate labor supply to zone j)}} \\
\end{align*}
\]

where the indicator \( I(\cdot, j, a_{od}) \) takes the value one if \( d = j \) (zero otherwise). Condition (4.48) becomes:

\[
\begin{align*}
Y^D = N \sum_{a_{ij} \in \mathcal{C}_{OD}} \left( \hat{a}_{ij} Y_{a_{ij}} \right). \quad (4.68)
\end{align*}
\]

Equation (4.46) is replaced by:

\[
\Psi_k = R \left\{ I(k \in J_W) \cdot K_{j}^{DF} (R, w_j, A_j | Q_j^S) + I(k \in J_R) \cdot K_{k}^{DF} (R, p_L | h_i^S) \right\} \quad (4.69)
\]

\[
\begin{align*}
\end{align*}\]

\[
\text{capital imports by firms} \\
+ N \Sigma_{a_{ij} \in \mathcal{C}_{OD}} \hat{a}_{ij} \times I(i = k, a) \times D_{Wa_{ij}} \times c_{aij} \quad \text{pecuniary commuting costs of those commuting from k}
\]

93
Now, we can solve the new system comprising of the blocks of equations (4.20), (4.24), (4.29), (4.43), (4.45), (4.47), (4.66) and (4.67), and the vectors of unknowns: \( p_{ij}, p_{Hi}, p, \, h_i^S, p_{Li}, w_j, Q_j^S \) and \( Y^S \). Following the price normalization of Section 4.3.1, we drop \( p \) (by fixing its price to one) and equation (4.47). To initiate the proposed algorithm, we make an initial guess over the above core vector and the expected travel times, pecuniary costs, toll costs for each OD-pair as well as the exogenous income, \( e \). Moreover, we guess the inclusive value of each OD-pair, \( \tilde{\omega}_{aij} \).

The new shortcut inner process consists of computing \( s_{aij}^*, T_{Faij}^* \) for each OD pair using (4.58) and (4.59), then inserting them into (4.56) to get \( y_{aij}^* \) and into (4.60) to get \( D_{Wa_{ij}}^* \). The approximation of the OD-pair indirect utility, \( V_{aij}^* \), is computed from (4.61). Summing together \( V_{aij}^* \) and the assumed inclusive values \( \tilde{\omega}_{aij} \) across OD pairs in \( \mathcal{C}_{OD} \) yields the denominator of (4.63). Computing the approximate choice probability \( \hat{\pi}_{aij} \) for each OD-pair is then straightforward. These probabilities are used to compute the right hand side of (4.66) and (4.68), and the left hand side of (4.67). The rest of the shortcut inner process, and the algorithm used to solve the core sub-system are similar to the ones described for the all-in-one solution, and can be found in detail in Appendix 4.B.

The economic equilibrium clears the markets for land, labor, housing and output and ensures zero profits. However, equations (4.38), (4.39), (4.40), (4.65) will not hold and OD-expected travel times, costs and toll expenditures will be incorrect. The following fixed point iteration solves this problem. To initiate it, we assume a travel time for each road and rail link. For each route, \( q \), we sum up the assumed travel time of each link composing it; this yields \( t_q^0 \). Because the corresponding pecuniary costs do not depend on congestion, we compute \( c_q \) directly from (4.34). The road tax per route, \( \tau_{Rq} \), is computed from (4.37).

We subsequently solve for the exact \( V_a^* \) of each alternative, \( i.e. \) we compute the Marshalian demand for space and leisure time in equations (4.6) and (4.7), labor supply in (4.8), demand for the composite (using (4.2)) and insert them in (4.12). With \( V_a^* \) calculated, the residual term \( \xi_q \) that approximates \( \tilde{\pi}_q \) can be computed for each route from (4.62). Inserting \( \xi_q \) in (4.64) yields the approximate route choice probability, \( \hat{\pi}_{q|aij} \). Now, the expected traffic load in each link can be calculated using (4.30). Then link travel times \( t_m^{(se)} \) can be updated using (4.31) and (4.32), and then summed up across routes to yield \( t_q^1 \). The process is repeated until the absolute value of \( t_q^{k+1} - t_q^k \) for every route falls short of a pre-specified tolerance.

Upon convergence, \( \hat{\pi}_{q|aij} \) is computed from (4.64); the inclusive values \( \tilde{\omega}_{aij} \) are updated from (4.65); and the OD time and cost expectations, \( i.e. \) \( t_{aij}, c_{aij} \) and \( \tau_{Ra_{ij}} \) are updated using:

\[
t_{aij} = \sum_{q \in \mathcal{Q}(aij)} \hat{\pi}_{q|aij} t_q,
\]  
(4.70)
\[ c_{a_{ij}} = \sum_{q \in Q(a_{ij})} \hat{\pi}_{q|a_{ij}} c_{q'} \]  \hspace{1cm} (4.71)

and:

\[ \tau_{Ra_{ij}} = \sum_{q \in Q(a_{ij})} \hat{\pi}_{q|a_{ij}} \tau_{Rq'} \]  \hspace{1cm} (4.72)

Finally, the tax revenue can be computed using:

\[ G_L = N \tau_L \sum_{a_{ij} \in C_{OD}} \left[ \hat{\pi}_{a_{ij}} D_{Wa_{ij}} w_j \right], \hspace{1cm} (4.73) \]

and:

\[ G_R = N \sum_{a_{ij} \in C_{OD}} \left[ \hat{\pi}_{a_{ij}} D_{Wa_{ij}} \left( \sum_{q \in Q(a_{ij})} \hat{\pi}_{q|a_{ij}} \tau_{Rq} \right) \right]. \hspace{1cm} (4.74) \]

Inserting (4.73) and (4.74) into (4.38) yields \( B \); \( B_e \) is given by (4.39) and (4.40) updates \( e \). The solution of the core subsystem and the fixed point iteration are repeated until the updates generated by the fixed point iteration fall short of the convergence criterion.

In general, the approach proposed here leads to a different equilibrium than the all-in-one approach. One way for the reader to confirm this is by comparing the value of time computed at the expected OD-pair times and costs, \( v_{a_{ij}} \), with the expected value of time from the all-in-one approach. This is:

\[ v_{a_{ij}} = \frac{w_j(1 - \tau_L) - c_{a_{ij}} - \tau_{Ra_{ij}}}{1 + t_{a_{ij}}} \]

\[ = \frac{w_j(1 - \tau_L) - \sum_{q \in Q(a_{ij})} \hat{\pi}_{q|a_{ij}} \left( c_{a_{ij}} + \tau_{Ra_{ij}} \right)}{1 + \sum_{q \in Q(a_{ij})} \hat{\pi}_{q|a_{ij}} t_{a_{ij}}} \]

\[ = \sum_{q \in Q(a_{ij})} \left\{ \hat{\pi}_{q|a_{ij}} \left( \frac{w_j(1 - \tau_L) - c_{a_{ij}} - \tau_{Ra_{ij}}}{1 + t_{a_{ij}}} \right) \right\} = E(v_{a}|i, j). \hspace{1cm} (4.75) \]

Despite this, the \textit{iterative-shortcut} approximation (algorithm) can provide good starting values to the \textit{all-in-one} algorithm; these values accelerate convergence to equilibrium. Because the excess demand functions of the iterative-shortcut approach differ from the corresponding all-in-one demands, switching from the \textit{iterative-shortcut} to the \textit{all-in-one} algorithm will cause a “jump” of
the endogenous vector from the iterative-shortcut equilibrium and will require additional iterations of the all-in-one algorithm to converge to the associated equilibrium.

In some cases, it is also possible to work with a model in which household optimization problem satisfies conditions 4.E.1 and 4.E.2 displayed in Appendix 4.E. Under these conditions, the iterative-shortcut approximates the all-in-one model perfectly; that is, the iterative-shortcut solution coincides with the solution from the all-in-one algorithm. For an iterative-shortcut specification that satisfies these conditions (and an illustration of the corresponding algorithm) the reader is referred to RELU-TRAN model by Anas and Liu (2007). Finally, the choice of iterative-shortcut specification does not affect the general comparative analysis between performance of the two algorithms, that follows in Section 4.3.3.

Figure 4.2. Schematic depiction of the iterative-shortcut algorithm.

4.3.3. Computational examples

We now explore the computational efficiency of the algorithms presented in sections 4.3.1 and 4.3.2, using four different network configurations (see Figure 4.3). The base configuration (upper left panel) comprises of 30 nodes and 90 links. Configuration BI adds fourteen links to
the existing ones without adding any additional nodes in the network. However, due to the substantial number of nodes and the location of the added links (i.e. they form a ring road around the city center), the resulting number of routes increases dramatically. Configuration B2 inserts twenty additional links while keeping the number of nodes fixed to thirty.

Table 4.1 illustrates the relation between the spatial configuration, the choice of solution method and computational time (expressed in milliseconds). For a given number of nodes (i.e. across the above three configurations) the number of excess demand equations and zero profit conditions remains constant: the size of the core vector changes in a one-on-one proportion to the number of newly introduced links. However, even a small number of new links in the network can have a dramatic effect upon the total number of routes. The additional computational burden does not stem from the number of new equilibrium conditions added (i.e. one for each link), but from the size of the underlying choice set (and therefore the number of computations needed to calculate each vector of the Jacobian matrix). This is why the all-in-one solution time is more or less proportional to the number of routes (which, in a large network, is roughly the number of alternatives) for any given number of nodes in the model. Appendix 4.D offers an improvised count of the operations required in one iteration of the all-in-one algorithm (Table 4.D.1 summarizes the count).

<table>
<thead>
<tr>
<th>Specification</th>
<th>nodes</th>
<th>links</th>
<th>routes</th>
<th>All-in-one solution time</th>
<th>IterativeShortcut solution time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base</td>
<td>30</td>
<td>90</td>
<td>6842</td>
<td>66367</td>
<td>51243</td>
</tr>
<tr>
<td>B1</td>
<td>30</td>
<td>104</td>
<td>129436</td>
<td>1521787</td>
<td>168062</td>
</tr>
<tr>
<td>B2</td>
<td>30</td>
<td>124</td>
<td>278208</td>
<td>3305657</td>
<td>280630</td>
</tr>
<tr>
<td>A1</td>
<td>50</td>
<td>148</td>
<td>169000</td>
<td>3254880</td>
<td>390103</td>
</tr>
</tbody>
</table>

Notes: Every route belongs to a unique alternative, but not vice versa. The total number of alternatives is given by \( r + n \) where \( r \) is the number of routes and \( n \) the number of nodes (= the number of non-commuting alternatives). Times are expressed in milliseconds. All computations were performed using an Intel Core i5 and 4GB of RAM.

On the other hand, the iterative-shortcut solution keeps the size of choice set and the Jacobian matrix fixed across the three aforementioned configurations. Therefore, the time needed for the economic/land-use model to converge remains constant. However, the number of operations performed in every fixed point iteration is roughly proportional to the number of routes (see Figure 4.2). This introduces the time increments reported for the iterative-shortcut solution in the last column of Table 4.1. Appendix 4.D repeats the count of operations for the iterative-shortcut algorithm.

---

\( ^93 \) To see why base configuration gives rise to a small number of routes relative to the configuration in B1, observe that in the former most OD pairs are characterized by a single route; this is because the base configuration characterizes a number of corridors with a tree structure.
The introduction of additional nodes in the network has a similar effect upon the computational time of the all-in-one solution. We illustrate this by returning to the base specification (upper left panel in Figure 4.3) and adding 20 additional nodes in a way such that the total number of resulting routes exceeds those of specification B1 but falls short of specification B2 (specification A1 in the lower right panel of Figure 4.3). In line with the earlier findings of this section, the all-in-one solution time is found to be longer than this of B1 but shorter of this needed to solve B2. The careful reader will also observe that the roughly linear relationship between the number of routes and the all-in-one solution time, that holds when keeping the number of nodes fixed, is now violated. Specification A1 gives rise to roughly 60% of the routes obtained by specification B2. However solving the former requires 98% of the time required for the solution of the latter. The incremental cost originates from the fact that the new nodes do not only add new links in the network; they also expand further the existing system of equations by adding new market clearing and zero profit conditions. With the help of Table 4.D.1 (and Figure 4.1) the reader can acquire a sense of the computational burden added in each loop of the algorithm.

Figure 4.3. Four different spatial configurations of the model: the base 30-node specification (upper left), a peripheral ring road extension (upper right), a further extension with expanded public transport (lower left) and the basic configuration with an increased, 50-node resolution (lower right). Straight lines represent undirected road links and bended curves undirected public transport links.
Finally, it is worth noting that the time increment recorded between specifications B2 and A1 in the iterative shortcut algorithm (i.e. 390103 - 280630 = 109500 milliseconds) is smaller than the respective increment recorded between specifications B2 and B1 (i.e. 280630 - 168062 = 112568 milliseconds). The explanation is that in the transition from B1 to B2 the additional burden originates from the addition of the 20 links that more than double the total number of routes, rendering the fixed point iteration slower. On the other hand, the transition from B2 to A1 involves a heavier land-use and transport model. But the same transition decreases the number of routes, offering a computational time discount through a faster fixed point iteration (compare the lower left and right panels of Figure 4.3).

4.4. Summary

Computable general equilibrium models of land use and transport (GELUT models) are complex and usually large scale, hence their accurate and fast solution by numerical methods is an important issue. This chapter shed light on the structure of an all-in-one GELUT model, in which the choices of residential and employment locations, route and mode are simultaneous. The mechanics of two competing approaches to solve the all-in-one model were explored. The first approach solves directly the above model for the general, stochastic user equilibrium (i.e. the simultaneous economic/land-use and transport equilibrium) with an all-in-one algorithm. The second approach attempts to approximate the all-in-one model with a different behavioral model (iterative-shortcut approximation model) in which the residential and employment location choices are detached from the mode and route choices. The approximation model is solved with an iterative-shortcut algorithm, i.e. by iterating between separate solutions of the land use and the transport equilibrium models. That reduces drastically the household’s choice set size, providing a relief from the computational burden of the all-in-one solution algorithm.

We show that the latter approach can provide tremendous time savings, which increase as model’s resolution becomes higher. The comparative analysis of the algorithms reveals that relative solution time (i.e. the proportion of all-in-one time needed by the iterative-shortcut approach) drops as new links are added in an existing set of nodes of a network. The exact drop depends on where these new links are added, since this determines how many new paths are generated in the network. In turn, the number of paths is shown to be the main driver of the time consumed by the all-in-one algorithm but not the key driver of the respective time consumed by the iterative-shortcut algorithm. The effect of adding new nodes (and therefore links) is more complex and depends not only on where the new links are added, but on the assumptions made for the newly added nodes. Finally, we identified the special conditions under which the solution of the iterative-shortcut approximation model represents the same equilibrium as the one obtained by the all-in-one algorithm. Even if these conditions do not hold, the solution of the
iterative-shortcut approximation model can provide good starting values for the all-in-one algorithm, accelerating the solution substantially.

Appendix 4.A: Notation

Table 4.A.1. Sets and network

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J )</td>
<td>the ordered set of all model zones</td>
</tr>
<tr>
<td>( J_R )</td>
<td>the subset of ( J ) that contains all residential zones</td>
</tr>
<tr>
<td>( J_R )</td>
<td>the number of elements in ( J_R )</td>
</tr>
<tr>
<td>( J_W )</td>
<td>the subset of ( J ) that contains all employment zones</td>
</tr>
<tr>
<td>( J_W )</td>
<td>the number of elements in ( J_W )</td>
</tr>
<tr>
<td>( i, j, k )</td>
<td>the indices used to denote arbitrary zones in sets ( J, J_R ) and ( J_W )</td>
</tr>
<tr>
<td>( C_{OD} )</td>
<td>the Cartesian product of sets ( J_R ) and ( J_W ), i.e. the set denoting all possible OD pairs</td>
</tr>
<tr>
<td>( a_{ij} )</td>
<td>an OD pair of ( C_{OD} ), i.e. a pair of the zone ( i ) in ( J_R ) and zone ( j ) in ( J_W )</td>
</tr>
<tr>
<td>( I(i, J_R) )</td>
<td>the indicator function that equals one if zone ( i ) belongs to ( J_R )</td>
</tr>
<tr>
<td>( I(j, J_W) )</td>
<td>the indicator function that equals one if zone ( j ) belongs to ( J_W )</td>
</tr>
<tr>
<td>( l_m^{(se)} )</td>
<td>the directed link of network type ( m ) that connects zone ( s ) to zone ( e )</td>
</tr>
<tr>
<td>( l_m^{(se)} )</td>
<td>the length of the directed link ( l_m^{(se)} )</td>
</tr>
<tr>
<td>( q )</td>
<td>an arbitrary path, i.e. an ordered list of links ( l_m^{(ij)} ) such that for each pair of consecutive links, ( l_m^{(se)} ) and ( l_{m'}^{(s'e')} ), it holds that ( e = s' )</td>
</tr>
<tr>
<td>( Q(a_{ij}) )</td>
<td>a set that contains all possible routes for the OD pair ( a_{ij} \in C_{OD} )</td>
</tr>
<tr>
<td>( a )</td>
<td>a choice alternative, a set that contains the OD pair ( a_{ij} ) and a route ( q \in Q(a_{ij}) ), i.e. ( a = {a_{ij}, q} )</td>
</tr>
<tr>
<td>( \mathcal{C} )</td>
<td>the choice set that contains all ( a ) in the model</td>
</tr>
<tr>
<td>( \pi_a )</td>
<td>the choice probability of an arbitrary alternative ( a )</td>
</tr>
<tr>
<td>( t_m^{(se)} )</td>
<td>travel time for the arbitrary directed link ( l_m^{(se)} )</td>
</tr>
<tr>
<td>( t_q )</td>
<td>the commuting time under the choice of alternative ( a = {a_{ij}, q} ), i.e. ( \sum_{t_m^{(se)} \in Q} t_{m}^{(se)} )</td>
</tr>
<tr>
<td>( \omega_{a_{ij}} )</td>
<td>the inclusive value of OD pair ( a_{ij} ), i.e. the expected maximum utility of the subset of ( \mathcal{C} ) that contains alternatives with OD pair ( a_{ij} )</td>
</tr>
<tr>
<td>( \bar{v}_{q</td>
<td>a_{ij}} )</td>
</tr>
</tbody>
</table>

Appendix 4.B: Proposed algorithms

4.B.1. General equilibrium in Section 4.3.1.

Inner process:

**input:** any current core vector \( \sigma = (p_I, p_H, t, h^S, p_L, w, Q^S, Y^S, e) \)
for each alternative \( a = \{a_{ij}, q\} \) in choice set \( C \):

Use the indices of origin zone \( i \), the destination zone \( j \) and the route \( q \) to retrieve \( p_{Hi} \), \( \ell_q \), \( w_j \) from \( \sigma \) and \( c_q \). Compute \( s_a^* \) using (4.6), \( T_{Fa}^* \) using (4.7) and \( D_{Wa}^* \) using (4.8). Insert \( s_a^* \) and \( D_{Wa}^* \) into (4.2) to solve for \( y_a^* \). Insert \( y_a^* \), \( s_a^* \) and \( T_{Fa}^* \) into the objective function to compute \( V_a^* \).

for each employment zone \( j \) in \( J_W \):

Use index \( j \) to retrieve \( p_{Ij} \), \( w_j \) and \( Q_j^S \). Compute \( K_j^{Df} \) and \( L_j^D \) from (4.18) and (4.19) respectively. Compute the difference between \( p_{Ij} \) and the right hand side of equation (4.20). Compute \( Q_j^D a \) using (4.23).

for each residential zone \( i \) in \( J_R \):

Use index \( i \) to retrieve \( p_{Hi} \), \( p_{Li} \) and \( h_i^S \). Compute \( K_i^{Da} \) and \( X_i^D \) from (4.27) and (4.28) respectively. Compute the difference between \( p_{Hi} \) and the right hand side of equation (4.29).

Compute the denominator of equation (4.15).

for each alternative \( a = \{a_{ij}, q\} \) in choice set \( C \): compute \( \pi_a \) using (4.15)

for each link \( l^{(se)}_m \) reset \( D^{(se)}_m = 0 \).

for each alternative \( a = \{a_{ij}, q\} \) in choice set \( C \):

if \( q \) non-empty then:

for each link \( l^{(se)}_m \) in \( q \):

Add \( \pi_a \times D_{Wa}^* \) to \( D^{(se)}_m \)

for each link \( l^{(se)}_m \): compute \( t^{(se)}_m \) using (4.31) or (4.32) depending on the nature of the link.

for each route \( q \) compute \( \hat{\ell}_q \) using (4.33).

Reset \( G_L = 0 \) and \( G_R = 0 \).

for each alternative \( a = \{a_{ij}, q\} \) in choice set \( C \):

Add the right hand side of (4.35) to \( G_L \).

if \( q \) non-empty then:

Add the right hand side of (4.36) in \( G_R \).

Compute \( B \) using (4.38) and \( B_\ell \) using (4.39).

Compute the right hand side of (4.42), (4.48) and the left hand side of (4.44) by adding across alternatives in the choice set.

Return: the core equations vector \( e(\sigma) \) consisting of (17), (26), (37), (38), (39), (40), (41), (42), (45) and its norm.
Jacobian of the core sub-system:

**input:** any current core vector \( \mathbf{o} = (p_I, p_H, t, h^S, p_L, w, Q^S, Y^S, e) \) and the corresponding vector \( e(\mathbf{o}) \)

**declare:** a square matrix \( \mathbf{J} \) of dimension equal to the number of elements in \( \mathbf{o} \)

for each element of vector \( \mathbf{o} \), i.e. \( o(i) \):
- **Define:** the perturbation vector \( \mathbf{d} \) with all elements equal to zero apart from the \( i \)-th, which is equal to \( \Delta o(i) \).
- **Perturb:** \( \mathbf{o}' = \mathbf{o} + \mathbf{d} \).
- **Compute:** \( e(\mathbf{o}') \) using the Inner process.
- **Calculate:** the numerical derivative vector \( \mathbf{f} = (e(o') - e(o))/\Delta o(i) \).
- **Replace:** the \( i \)-th column of the Jacobian matrix \( \mathbf{J} \) with \( \mathbf{f} \).

Solution algorithm:

**Initiation:** guess initial staring values for the set of endogenous vectors (e.g. all prices equal to one, all stocks zero)

**Specify:** tolerance \( \gamma \)

**Set** iteration \( k = 0 \)

**Compute:** the vector \( e(\mathbf{o}_0) \)

while \( (\|e\| > \gamma) \)
{
    Compute the Jacobian matrix \( \mathbf{J} \) through Method 5.
    Compute the inverse Jacobian, \( \mathbf{J}^{-1} \).
    Compute the proposed correction, i.e. \( \mathbf{J}^{-1}e \).
    Update \( \mathbf{o}_{k+1} = \mathbf{o}_k - \delta \mathbf{J}^{-1}e_k \) using a constant \( \delta \) based on a line search.
    Compute \( e(\mathbf{o}_{k+1}) \)
}

4.B.2. General equilibrium in Section 4.3.2

Shortcut inner process:

**input:** any current core vector \( \mathbf{o} = (p_I, p_H, h^S, p_L, w, Q^S, Y^S) \) and \( \omega_{aij}, e \) which are determined in the fixed point iteration (see below).

- for each OD pair \( a_{ij} \) in set \( \mathcal{C}_{OD} \):
Use the indices of origin zone $i$, the destination zone $j$ to retrieve $p_{Hi}$, $w_j$ from $\sigma$. Compute $s^*_{aij}$ using (4.58), $T^*_{Fa_{ij}}$ using (4.59) and $D^*_{Wa_{ij}}$ using (4.60). Insert $s^*_{aij}$ and $T^*_{Fa_{ij}}$ into (4.56) to solve for $y^*_{aij}$. Insert $y^*_{aij}$, $s^*_{aij}$ and $T^*_{Fa_{ij}}$ into (4.61) to compute $V^*_{aij}$.

- for each employment zone $j$ in $J_W$:
  Use index $j$ to retrieve $p_{lj}$, $w_j$ and $Q^S_j$. Compute $K^{Df}_j$ and $L^D_j$ from (4.18) and (4.19) respectively. Compute the difference between $p_{lj}$ and the right hand side of equation (4.20). Compute $Q^{DA}_j$ using (4.23). Compute the excess demand for each intermediate in (4.45).

- for each residential zone $i$ in $J_R$:
  Use index $i$ to retrieve $p_{Hi}$, $p_{Li}$ and $h^S_i$. Compute $K^{Dd}_i$ and $X^D_i$ from (4.27) and (4.28) respectively. Compute the difference between $p_{Hi}$ and the right hand side of equation (4.29). Compute the difference between $\bar{X}_i$ and $X^D_i$ in (4.43).

- Compute the denominator of equation (4.63).
- for each alternative $a_{ij}$ in set $C_{OD}$: compute $\hat{a}_{aij}$ using (4.63).

- for each employment zone $j$ in $J_W$:
  using $\hat{a}_{aij}$ and $D^*_{Wa_{ij}}$, compute the aggregate labor supply to $j$, as it is defined in (4.67).

- for each residential zone $i$ in $J_R$:
  using $\hat{a}_{aij}$ and $s^*_{aij}$, compute the aggregate residential space demand in $i$, as it is defined in (4.66).
  using $\hat{a}_{aij}$ and $y^*_{aij}$, compute the aggregate demand for the numéraire in $i$, as it is defined in (4.68).

- sum up the aggregate demands for the numéraire to get $Y^D$. Compute $p(Y^S - Y^D)$, i.e the left hand side of (4.47).
- for each zone $k$ in $J$: using $D_{Wa_{kj}}$, $c_{akj}$, $K^D_{k}$, $K^d_k$ compute the total value of exports, $\Psi_k$, in (4.69).
- Sum up $\Psi_k$ and compute (4.47).

\textbf{Return} the core equations vector $e(\sigma)$ consisting of the differences between the left and right hand side of equations: (4.20), (4.29), (4.66), (4.43), (4.67), (4.45), (4.47) and its norm.

The Jacobian and the solution algorithm are those proposed earlier, adjusted for the core vector $\sigma$. 

Fixed point iteration and the stochastic user equilibrium:

**Initiation:** make a guess of $t_{m}^{(se)}$ and add up to get $t_{q}^{0}$ for every possible route.

The $k$-th iteration is repeated as long as $\|t_{q}^{k} - t_{q}^{k-1}\|$ exceeds the specified tolerance. It consists of the following:

- for each alternative $a = \{a_{ij}, q\}$ in choice set $C$:
  - Use the indices of origin zone $i$, the destination zone $j$ and the route $q$ to retrieve $p_{Hl}$, $\tau_{a_{ij}}^{k-1}$, $w_{j}$ from $\sigma$ and $c_{q}$ (from wherever it is stored). Compute $s_{a}^{*}$ using (4.6), $T_{Fa}^{*}$ using (4.7) and $D_{Wa}^{*}$ using (4.8). Insert $s_{a}^{*}$ and $D_{Wa}^{*}$ into (4.2) to solve for $y_{a}^{*}$. Insert $y_{a}^{*}$, $s_{a}^{*}$ and $T_{Fa}^{*}$ into the objective to compute $V_{a}^{*}$. Calculate the deviation terms $\xi_{q}$ from (4.62).

- for each OD pair $a_{ij}$ in set $C_{OD}$: compute the denominator of (4.64).
- for each OD pair $a_{ij}$ in set $C_{OD}$:
  - for each route $q$ in $Q(a_{ij})$:
    - compute $\pi_{q|a_{ij}}$ using (4.64).
  - Compute the demand for each link using $D_{Wa}^{*}$, the logit probability decomposition and the formula in (4.30).
  - Compute the travel time for each link using (4.31) or (4.32).
  - Update each route travel time, $t_{q}^{k-1}$, using the right hand side of (4.33).

Upon convergence: update $\hat{\omega}_{aij}$ by computing (4.65) and $e$ by computing (4.35), (4.36), (4.38), (4.39) and (4.40). Compute $t_{aij}$, $c_{aij}$ and $\tau_{Ra_{ij}}$ from (4.70), (4.71) and (4.72).

**Appendix 4.C: A note on the correct use of numéraire**

A special version of the model regards the case in which economic and transport activity is completely partitioned into a number of zone groups across which no trade of goods or cross commuting takes place. Despite this case is purely theoretical, we briefly discuss it here to further illustrate the importance behind the correct use of numéraire and the appropriate closure. Furthermore, this polar case sheds light on how spatial markets interact with each other. This can provide insights on the efficiency of Gauss-Jacobi algorithms (that sequentially solve for subsystems of the one described in 4.3.1).

To keep the illustration close to this of a textbook, we consider four zones: two residential and two employment, i.e. $J = \{A_{R}, B_{R}, A_{W}, B_{W}\}$, $J_{R} = \{A_{R}, B_{R}\}$ and $J_{W} = \{A_{W}, B_{W}\}$. Thus, $J_{R} \cap J_{W} = \emptyset$ and mixed land-used is prohibited. The good produced in $A_{W}$ is consumed as a final good in $A_{R}$ and the good produced in $B_{W}$ is consumed as a final good in $B_{R}$.
Furthermore, there is no cross commuting: workers who reside in zone $A_R$ commute to $A_W$ and workers who reside in zone $B_R$ commute to $B_W$. Therefore, residential and employment choice location are switched-off. There is no public transport; commuting takes place through two independent road links ($l_c^{02}, l_c^{13}$, where superscripts refer to the zones, as they are ordered in set $J$).

Because there are two endogenous economies in this model, i.e. the economic activity between the zonal subset $J_A = \{A_R, A_W\}$ and ROW is independent of this between the zonal subset $J_B = \{B_R, B_W\}$ and ROW, model closure demands two conditions instead of one. Each of them postulates that the value of imports from ROW is equal to the value of foregone consumption in the region. Therefore, condition (4.47) must be replaced by the following two separate closing conditions:

$$p_0(Y_0^S - Y_0^D) = \Psi_0,$$  \hspace{1cm} (4.76)

$$p_2(Y_2^S - Y_2^D) = \Psi_2,$$  \hspace{1cm} (4.77)

where $p_0$ and $p_2$ are the local price indices that are both normalized. Thus both (4.76) and (4.77) are excluded the core equations (together with prices $p_0, p_2$) and are used as convergence and specification tests.

To see why closure takes this form, consider again the virtual trader, who makes zero profits by borrowing an amount of $Y_0$ of value equal to $p_0(Y_0^S - Y_0^D) + p_2(Y_2^S - Y_2^D)$ from the assembly industry located in zone $A_R$. Then, the trader spends this borrowed amount to cover the total value of imports across both regions ($A_R$ and $B_R$), i.e. to buy imports of value $\Psi_0 + \Psi_2$ from the rest of the world (ROW). After repaying part of its debt by returning imports of value $\Psi_0$ to $A_R$, the virtual trader is left with imports (of value $\Psi_2$) which he sells to $B_R$ to receive output $Y_2$ of equal value. However: i) there is no trade between $A_R$ and $B_R$ and ii) ROW has not been modelled explicitly (so that imports can play the role of numéraire). Thus, the acquired output $Y_2$ cannot be sold to $A_R$ in order to repay the remaining debt to it: the model does not close. The above model can be solved more efficiently with an algorithm that computes a separate equilibrium for each region.

**Appendix 4.D: Comparative algorithm analysis**

This appendix provides a quick count of the number of operations performed by the algorithms discussed in the chapter. Because the size of choice set in the all-in-one solution equals the sum of the number of routes and the number of no-commuting alternatives, that is: $N_c = N_p(J, N^C) + N(J_R \cap J_W)$. Computations in (4.6)-(4.12) are repeated in total $N_c$ times in each computation of the core system. On the other hand, the iterative shortcut approach reduces the size of the choice set in the economic model to $(J_R \times J_W)$, i.e. the number of origin-destination pairs that can be constructed from the sets of residential and employment zones. On top of that, there are $(3 \cdot J_W + 3 \cdot J_R + 3)$ firm behavioral operations executed (these are computations of conditional factor
demands and marginal costs that are used to form the zero profit, market clearing conditions and the balanced budget equations).

Next, the traffic assignment requires on average $N_P(J, N^C_\ell) \cdot \bar{\tau}_P(J, N^C_\ell)$ computations to sum up the demand $D^{(se)}_m$ for any given link in (4.30). That is, for each link $l^{(se)}_m$ in the network, the algorithm accesses each of the $N_P(J, N^C_\ell)$ commuting routes of the network and checks whether the sequence of links in that route, $q$, contains the link $l^{(se)}_m$. The average number of links $l^{(od)}_m$ in a route $q$ (and therefore of the number of checks) is a function of the network specification, that is: $\bar{\tau}_P(J, N^C_\ell)$. Repeating across the $(N^C_\ell + N^R_\ell)$ links yields:

$$\left(N^C_\ell + N^R_\ell\right) \cdot N_P(J, N^C_\ell) \cdot \bar{\tau}_P(J, N^C_\ell).$$

(4.78)

checks. The network loading requires the algorithm to perform additional $(N^C_\ell + N^R_\ell)$ computations. The inner processes differ: the iterative-shortcut algorithm removes traffic assignment and network loading from the inner process (whose computational cost is transferred to the fixed point iteration) and reduces the choice set operations. Computing the Jacobian matrix requires repeating the inner process $(3 \cdot J_W + 3 \cdot J_R + 3 + N^C_\ell)$ times in the all-in-one case and $(3 \cdot J_W + 3 \cdot J_R + 3)$ in the iterative-shortcut, i.e. as many as the number of core equations (and the size of the core vector) in each case.

**Table 4.D.1. Counting operations in the all-in-one algorithm**

<table>
<thead>
<tr>
<th>Code snippet</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Choice set operations</td>
<td>$Q^CS_1 = N_P(J, N^C_\ell) + N(J_R \cap J_W)$</td>
</tr>
<tr>
<td>Firm operations</td>
<td>$Q^{FB}_1 = 3 \cdot J_W + 3 \cdot J_R + 3$</td>
</tr>
<tr>
<td>Traffic assignment</td>
<td>$Q^{TA}<em>1 = \left( N^C</em>\ell + N^R_\ell \right) \cdot N_P(J, N^C_\ell) \cdot \bar{\tau}<em>P(J, N^C</em>\ell)$</td>
</tr>
<tr>
<td>Network loading</td>
<td>$Q^{NL}<em>1 = \left( N^C</em>\ell + N^R_\ell \right)$</td>
</tr>
<tr>
<td>Inner process</td>
<td>$Q^{IP}_1 = (Q^{CS}_1 + Q^{FB}_1 + Q^{TA}_1 + Q^{NL}_1)$</td>
</tr>
<tr>
<td>Jacobian</td>
<td>$Q^{FB}<em>1 = (3 \cdot J_W + 3 \cdot J_R + 3 + N^C</em>\ell) \cdot Q^{IP}_1$</td>
</tr>
</tbody>
</table>
Table 4.D.2. Counting operations in the iterative-shortcut algorithm

<table>
<thead>
<tr>
<th>Code snippet</th>
<th>Choice set operations $Q_2^{CS} = N(J_R \times J_W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm operations</td>
<td>$Q_2^{FB} = 3 \cdot J_W + 3 \cdot J_R + 3$</td>
</tr>
<tr>
<td>Traffic assignment</td>
<td>$Q_2^{TA} = (N_C^C + N_R^C) \cdot N_p(I,J_N^C) \cdot \bar{x}_p(I,J_N^C)$</td>
</tr>
<tr>
<td>Network loading</td>
<td>$Q_2^{NL} = (N_C^C + N_R^C)$</td>
</tr>
<tr>
<td>Inner process</td>
<td>$Q_2^{IP} = (Q_2^{CS} + Q_2^{FB})$</td>
</tr>
<tr>
<td>Jacobian</td>
<td>$Q_2^{FB} = (3 \cdot J_W + 3 \cdot J_R + 3) \cdot Q_2^{IP}$</td>
</tr>
<tr>
<td>Fixed point iteration</td>
<td>$Q_2^{FP} = (Q_2^{TA} + Q_2^{NL})$</td>
</tr>
</tbody>
</table>

Appendix 4.E: Coincidence of approaches

Here we show that, under certain conditions, the all-in-one and the iterative-shortcut algorithm reach the same equilibrium. Denote by $x_{ijr}(p_{ij}, \theta_r)$ the arbitrary endogenous choice variable (e.g. residential demand, labor supply, consumption etc.), by $p_{ij}$ the prices varying at the nest level and by $\theta_r$ the route characteristics. Assume that:

$$x_{ijr}(p_{ij}, \theta_r) = x_{ij}(p_{ij}) + x_r(\theta_r).$$ (4.E.1)

That is, demand can be decomposed in a part varying with OD-pair prices and that the indirect utility, $V$, is such that:

$$V(x_{ijr}(p_{ij}, \theta_r)) = V_1(x_{ij}(p_{ij})) + V_2(x_r(\theta_r)).$$ (4.E.2)

Under E.2, it holds that:

$$P_{ijr} = P_{ij}P_{r|ij}.$$ (4.E.3)

**Proposition.** Under assumptions 4.E.1 and 4.E.2, the aggregate demand functions, and therefore the excess demands of the two approaches coincide.

**Proof.** The expected aggregate demand for an arbitrary good $x$ under the all-in-one approach is:

$$E(X) = \sum_{ijr} (P_{ijr}x_{ijr}(p_{ij}, \theta_r)) = \sum_{ijr} [(P_{ij}P_{r|ij}) (x_{ij}(p_{ij}) + x_r(\theta_r))]$$ (4.E.4)
\[
\sum_{ijr} \left( (P_{ij}P_{r|ij}) \cdot x_{ij}(p_{ij}) + (P_{ij}P_{r|ij}) \cdot x_r(\theta_r) \right) = \\
\sum_{ijr} \left( (P_{ij}P_{r|ij}) \cdot x_{ij}(p_{ij}) \right) + \sum_{ijr} \left( (P_{ij}P_{r|ij}) \cdot x_r(\theta_r) \right) = \\
\sum_i \sum_r \left( (P_{ij}P_{r|ij}) \cdot x_{ij}(p_{ij}) \right) + \sum_r \sum_{ij} \left( (P_{ij}P_{r|ij}) \cdot x_r(\theta_r) \right) = \\
\sum_{ij} P_{ij} \cdot x_{ij}(p_{ij}) \sum_r P_{r|ij} + \sum_r P_{r|ij} \cdot x_r(\theta_r) \sum_{ij} P_{ij} = \\
\sum_{ij} P_{ij} \cdot x_{ij}(p_{ij}) + \sum_r P_{r|ij} \cdot x_r(\theta_r)
\]

which is simply the expected aggregate iterative-shortcut demand. But because firm behavior is identical in the two formulations, excess demands functions, and therefore equilibria, will coincide. For a model that satisfies 4.E.1 and 4.E.2, the reader is referred to RELU-TRAN by Anas and Liu (2007).

**Appendix 4.F: Derivation of closed-form indirect utility**

Using (4.3), (4.6) and (4.7) someone can derive:

\[
p_{HI}s^*_a = p_{HI} \left( \frac{p_{HI} \pi_0}{\alpha \gamma \pi_1} \right)^{\frac{1}{\gamma - 1}} \left( \frac{\alpha v_a}{\beta p_{HI}} \right)^{\frac{\beta y}{\gamma - 1}} = p_{HI} \left( \frac{\alpha v_a}{\beta \pi_1} \right)^{\frac{1}{\gamma - 1}} \left( \frac{\alpha v_a}{\beta} \right)^{\frac{\beta y}{\gamma - 1}} \tag{4.F.1}
\]

and:

\[
v_a T^*_F = v_a \left( \frac{p_{HI} \pi_0}{\alpha \gamma \pi_1} \right)^{\frac{1}{\gamma - 1}} \left( \frac{\alpha v_a}{\beta p_{HI}} \right)^{\frac{\beta y}{\gamma - 1}} = v_a \left( \frac{\alpha v_a}{\beta \pi_1} \right)^{\frac{1}{\gamma - 1}} \left( \frac{\alpha v_a}{\beta} \right)^{\frac{\beta y}{\gamma - 1}}. \tag{4.F.2}
\]

Inserting 4.F.1 and 4.F.2 in (4.3) yields the optimal consumption:

\[
y_a^* = e + v_a T - (p_{HI} s^*_a + v_a T^*_F) = \]

\[
e + v_a T \left( \frac{\pi_0}{\alpha \gamma \pi_1} \right)^{\frac{1}{\gamma - 1}} \left( \frac{\alpha v_a}{\beta} \right)^{\frac{\beta y}{\gamma - 1}} \alpha^{-1}. \tag{4.F.3}
\]
Multiplying it by \( \pi_0 \), then, yields:

\[
\pi_0 y^*_a = \pi_0 (e + v_a T) - v_a^{y-1} p_{Hi}^{\frac{\beta y}{\alpha y - \gamma}} \frac{\alpha y - \gamma}{\gamma - 1 - \gamma} \alpha^{y-1} \frac{\beta y}{\gamma - 1 - \gamma}.
\] (4.F.4)

Moreover it holds that:

\[
(s^*_a)^\alpha = \left(\frac{p_{Hi} \pi_0}{\alpha y \pi_1}\right)^{\frac{\alpha y - \gamma}{\gamma - 1}} \frac{\alpha v_a}{\beta p_{Hi}} \frac{\alpha y - \gamma}{\gamma - 1}.
\] (4.F.5)

and

\[
(T^*_F a)^\beta = \left(\frac{p_{Hi} \pi_0}{\alpha y \pi_1}\right)^{\frac{\beta y}{\gamma - 1}} \frac{\alpha v_a}{\beta p_{Hi}} \frac{\beta y - \gamma y + \beta}{\gamma - 1}.
\] (4.F.6)

so that:

\[
(s^*_a)^\alpha (T^*_F a)^\beta = \left(\frac{p_{Hi} \pi_0}{\alpha y \pi_1}\right)^{\frac{1}{\gamma - 1}} \frac{\alpha v_a}{\beta p_{Hi}} \frac{\alpha y - \gamma}{\gamma - 1}
\]

\[
[(s^*_a)^\alpha (T^*_F a)^\beta]^\gamma = \left(\frac{p_{Hi} \pi_0}{\alpha y \pi_1}\right)^{\frac{\gamma}{\gamma - 1}} \frac{\alpha v_a}{\beta p_{Hi}} \frac{\alpha y - \gamma}{\gamma - 1}
\]

\[
[(s^*_a)^\alpha (T^*_F a)^\beta]^\gamma = \left(\frac{p_{Hi} \pi_0}{\alpha y \pi_1}\right)^{\frac{\gamma}{\gamma - 1}} \frac{\alpha v_a}{\beta p_{Hi}} \frac{\alpha y - \gamma}{\gamma - 1}
\]

\[
\pi_1[(s^*_a)^\alpha (T^*_F a)^\beta]^\gamma = v_a^{\gamma - 1} p_{Hi}^{\gamma - 1} \pi_0^{\gamma - 1} \pi_1^{\gamma - 1} \alpha^{\frac{\gamma - 1}{\gamma - 1}} \frac{\beta y}{\gamma - 1} \frac{\gamma - 1}{\gamma - 1}.
\] (4.F.7)

Finally, inserting (4.F.4) and (4.F.7) into the objective function yields:

\[
V^*_a = z_a + \pi_0 y^*_a + \pi_1 (s^*_a (T^*_F a)^\gamma) =
\]

\[
= z_a + \pi_0 (e + v_a T)
\]

\[
+ v_a^{\frac{\beta y}{\gamma - 1}} p_{Hi}^{\frac{\alpha y - \gamma}{\gamma - 1} \pi_0^{\gamma - 1} \pi_1^{\gamma - 1} \gamma} \left\{ \frac{\alpha v_a}{\beta p_{Hi}} \frac{\beta y - \gamma y + \beta}{\gamma - 1} \right\}.
\] (4.F.8)