Chapter 3

Communication and Coordination in Social Networks: Actions as Signaling Device

3.1 Introduction

In this chapter we consider an \( n \)-person multi-period game with an underlying social network, where a directed link expresses the observability of actions and payoff functions of the agent from whom the link starts, by the agent at the end of the link. Two actions, the status quo and an irreversible alternative, are possible. In every period an agent is supposed to choose one from the two actions if her choice in the previous period is the status quo. The payoff function of each agent depends on the threshold of the agent, i.e., the minimum number of agents who choose the alternative that is needed for her to be satisfied in order to join. Heterogeneity is considered by allowing for different thresholds between agents. An agent who chooses the alternative action obtains a higher payoff than staying with the status quo if her threshold is met, otherwise she must pay a penalty. The aim of this chapter is to investigate how coordination on the alternative action depends on the other parameters, i.e. thresholds, observability structure (the communication network\(^1\)), and penalty for choosing the alternative action without enough participants. For convenience we simply say coordination instead of coordination on the alternative action for short. We particularly focus on the role of acting at an early stage as signaling device in achieving a broader coordination through some illustrative examples. This signaling effect may encourage the emergence of coordination. It is shown that for some network structures, a lower threshold can be an ob-

\(^1\)It will be made clear later that by communication we do not mean that agents can pass information of others to a neighbor by telling her.
stacle of coordination. We also show that a restriction in observability, i.e., reducing links from the social network, may help attain full coordination. These seemingly paradoxical results are not obtained in previous studies which only consider a game with single-period simultaneous moves or homogeneous payoffs (see, for instance, Chwe (1999, 2000)). The penalty of choosing the alternative without enough participants is another important parameter of this game. It is also shown that, in behavior strategies, the probability of choosing the alternative action can be an increasing function of the penalty when the penalty is low enough. For a large penalty there is no coordination at all.

Our motivation is to provide a direct extension to Chwe (1999, 2000), in which a similar model is considered but essentially only in a single-period simultaneous move game. Our setting differs in the sense that in our model, payoffs are only paid when the game has finished, but not in each period or at the moment that an agent makes a decision. This assumption is essential to allow “strategic” play of the alternative action of an agent in order to pass signals to trigger actions of her neighbors. In the single-period case, the results of Chwe (1999, 2000) suggest that the lower the thresholds are, or the more the links are in the network, the easier full coordination can be achieved. In our multi-period model, it will be seen that this monotonic property is not preserved any more.

This chapter specially contributes to dynamic coordination problems in social learning studies. The research of social learning, including Bayesian and non-Bayesian, has a vast literature. One trend can be categorized as sequential learning models, started by Banerjee (1992), Bikhchandani, Hirshleifer, and Welch (1992), and Welch (1992), then followed and extended by Acemoglu et al. (2011), Banerjee and Fudenberg (2004), and Smith and Sørensen (2000). In their models, players are assumed to take action sequentially with a predetermined order, with each of them having a private signal about the unknown state of nature. Bayesian learning leads to herding behavior, which means players follow the information observed from their predecessors rather than that from their own signals. Another direction is learning in social networks, initiated by the seminal work of Bala and Goyal (1998). They present a network model of repeated game where observability of actions and payoffs of other players is restricted to direct neighbors of each player. Payoffs are random variables and players only observe the realization in each period. A bounded-Bayesian learning algorithm is considered where players only use the information of their neighbors, but do not make inferences about the neighbors of neighbors. Followers of Bala and Goyal (1998) include Gale and Kariv (2003), Mueller-Frank (2013), and Rosenberg, Solan, and Vieille (2009), among others. Although these papers use different assumptions: rational or bounded rational agents, observability of payoffs, random or deterministic payoff functions, etc.; they coincide in assuming that agents are myopic and maximize their short-run payoffs, where
the payoffs do not depend on actions of other agents. In other words, they all assume pure informational externalities. Of interest is convergence in opinion and action in the long run. Other non-Bayesian social learning studies, often called social influence models, include Acemoglu, Ozdaglar, and ParandehGheibi (2010), DeGroot (1974), DeMarzo, Vayanos, and Zwiebel (2003), Golub and Jackson (2010), and Jadbabaie et al. (2012). In these papers agents are assumed to average the observations obtained from their neighbors rather than update beliefs based on Bayes’ rule. Acemoglu and Ozdaglar (2011) provides an updated overview of social learning literature.

Our research is distinct from the above literature in the following aspects: 1) agents are real interactive: payoffs depend on their own actions as well as their neighbors’; 2) payoff is paid when the game is finished; 3) rational learning is considered. These characterizations emphasize that our model is not a repeated game but a multi-period game where players simultaneously choose whether to switch to the irreversible alternative in each period. Other recent studies of coordination games include Acocella et al. (2014), Dasgupta (2007), Dasgupta, Steiner, and Stewart (2012), Hagenbach and Koessler (2010), Kováč and Steiner (2013), and Sákovics and Steiner (2012), among others.

The rest of the chapter is organized as follows. In Section 3.2 we provide a general definition of the game. Case studies of equilibria in pure and behavior strategies on games with a small number of agents are given in Section 3.3 and 3.4, respectively. Section 3.5 concludes this chapter.

### 3.2 The model

We consider an $n$-person multi-period game. Each agent can take an action from $\{S, R\}$, where $S$ means staying with the status quo and $R$ represents the irreversible alternative option, which could mean voting for non-confidence, switching to a new technology, or joining a revolution, etc. The payoff of each agent depends on her threshold, her own choice, and the number of agents who take action $R$ (including the agent herself). If we denote the threshold of agent $i$ by by $\theta_i$, the payoff of $i$ is then given by

$$
\begin{cases}
  1 & \text{if } i \text{ takes action } R \text{ and } \#\{\text{agents who take action } R\} \geq \theta_i, \\
  0 & \text{if } i \text{ takes action } S, \\
  -P & \text{if } i \text{ takes action } R \text{ and } \#\{\text{agents who take action } R\} < \theta_i.
\end{cases}
$$

\footnote{Usually a repeated game contains repetition of a stage-game.}
Here, $P > 0$ is the penalty of choosing the alternative action without enough coordinators that do the same. We assume that the possible threshold of each agent is in the set \( \{m, \ldots, n + 1\} \) where $m$ is the lowest threshold that is common knowledge to all agents. The number $m$ is given as a parameter of the game, and can be taken from 1 to $n + 1$. In many real situations, for instance in the decision of whether to buy a new iPhone where the status quo is not to buy, it is natural to take $m = 1$ since one can be satisfied by enjoying the functions in the new iPhone. If the decision is whether to make a telephone contract where everyone starts with no contract, then $m$ must be 2 to make the game meaningful, since it makes no extra utility if you are the only one who has a telephone. Here threshold 1 represents obvious initiators who will always be better off by choosing action $R$. Agents with threshold $n + 1$ are people who will never choose the alternative.

Agents are connected in a directed network. A link from $i$ to $j$ represents the observability of action and threshold of agent $i$ by agent $j$. In other words, links represent flow direction of information. The neighbors of $i$ is thus the set of agents with a link directed to $i$. We assume that the network structure is known to all agents.

In contrast to Chwe (1999, 2000), we consider a multi-period game instead of a single-period one. In each period, all agents simultaneously decide which action to take. Once an agent chooses action $R$, it is not possible to switch back to $S$ again. Payoffs are paid when the game has finished. The number of periods is finite and denoted by $T$. To make the analysis simple, we assume that if an agent is indifferent in expected payoff of playing $R$ in different periods, she will act as early as possible. Although the value of $T$ is arbitrary, we can safely assume that $T$ is no more than the diameter of the network plus one, since it is the maximum time that will be taken for information to reach all agents in the network. Thus after period $T$, no agent will have reasons to change her choice. Agents are fully rational.

This is a game with incomplete information, which can be represented as a Bayesian game with imperfect information in Harsanyi's sense (Harsanyi 1967, 1968a,b)). A type of an agent is a subset of the space of all possible threshold vectors, and all types of an agent constitute a partition of the threshold vector space. All agents are assumed to have a uniform prior belief over the type space which is common knowledge, and update their beliefs after the move of nature based on their own types.$^3$

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$^3$Beliefs about the types of other players can be expressed as beliefs of threshold vectors, which usually makes the representation of the game simpler since probabilities of those infeasible combinations of types are zero and can be disregarded.
3.3 A case study of equilibria in pure strategies

In this section, we discuss the existence of equilibria with full coordination of action $R$ at the end of a game. In particular, we focus on equilibria in which some agents behave as initiators and others follow. We employ (weak) perfect Bayesian equilibrium (PBE)$^4$ as the equilibrium concept. We note that although we do not do general analysis on the class of games defined in the previous section, the interesting properties that are unique to this type of multi-period game compared to the corresponding single-period game is shown with sufficient evidence.

As discussed in Chwe (1999), there may exist multiple pure strategy Nash equilibria for the single-period version game. For example, in the two-person game with a threshold vector $(2, 2)$ where the two agents are mutually observable, both strategy profiles $(S, S)$ and $(R, R)$ are Nash equilibria. It can be seen that for multi-period version of this game, when $m > 1$ and $P$ is sufficiently large, playing $S$ in all periods for both agents is always an equilibrium. This leads to an equilibrium selection problem when multiple equilibria exist. In this section, we assume that, whenever there are multiple equilibria in pure strategies, those Pareto inefficient ones are not played. In the single-period mutually observable two-person game with threshold vector $(2, 2)$, this means the equilibrium strategy profile $(S, S)$ is ruled out.

3.3.1 Some three-person games

In this subsection we assume the penalty $P$ is sufficiently large in order to restrict our attention on the effects of network structure and thresholds. We conveniently take $P \rightarrow \infty$, which should not be interpreted as asymptotic analysis. We start with a game with three agents connected in a line network.

**Game 3.1.** Three agents are connected in a line as depicted in Figure 3.1, where agent $B$ can observe the thresholds and actions of both agents $A$ and $C$, but $A$ and $C$ can only observe those of agent $B$. The thresholds are $\theta_A = \theta_B = \theta_C = 3$, and the minimum threshold is assumed to be $m = 3$. The number of periods is $T = 2$.

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$^4$Here we refer to the definition in Mas-Colell, Whinston, and Green (1995).
The extensive form representation of this game is sketched in Figure 3.2. It should be noted that Figure 3.2 covers a set of games that the threshold vector $\theta$ satisfies $\theta_i \in \{3, 4\}$ for $i \in \{A, B, C\}$. In case of Game 3.1, all the three agents know that the threshold of agent $B$ is 3 after the move of nature, therefore the posterior probabilities that nature chooses nodes 343, 344, 443, 444 are zero, which means the right half part of the game tree can be ignored.

Obviously, the full vector $\theta$ is only known to $B$. $A$ and $C$ cannot observe each other’s threshold and action. To agent $A$ (agent $C$), the threshold of $C$ (agent $A$) can be 3 or 4 with equal probability. The following strategy profile together with an appropriate belief system constitute a PBE.\(^5\)

- Strategy of agent $A$:
  
  **Period 1** Choose $S$.
  
  **Period 2** Choose $R$ if she has chosen $R$ in period 1, or if agent $B$ has chosen $R$ in period 1 and $\theta_A = 3$, otherwise choose $S$.

- Strategy of agent $B$:

  **Period 1** Choose $R$ if $\theta_A = \theta_B = \theta_C = 3$, otherwise choose $S$.

  **Period 2** Choose $R$ if she has chosen $R$ in period 1, otherwise choose $S$.

- Strategy of agent $C$:

  **Period 1** Choose $S$.

  **Period 2** Choose $R$ if she has chosen $R$ in period 1, or if agent $B$ has chosen $R$ in period 1 and $\theta_C = 3$, otherwise choose $S$.

This strategy profile is a PBE of Game 3.1, which can be verified by evaluating the expected payoffs in the extensive form game. Following this equilibrium strategy profile, a full coordination on switching to $R$ at $T = 2$ is achieved. The more informed agent $B$ initiates the revolting in period 1, which is followed by the two less informed agents $A$ and $C$ in the next period. Here choosing action $R$ in period 1 by $B$ signals to $A$ that the threshold of $C$ is $\theta_C = 3$ (and reversely $\theta_A = 3$ to $C$). In contrast, in the single-period version of this game, strategy profile $(R, R, R)$ is not an equilibrium. This is because, e.g. for agent $A$, with probability $1/2$ the threshold $\theta_C$ is 4 and hence $C$ does not revolt, resulting in a negative payoff $-P$ for $A$. Following the assumption $P \to \infty$, the expected payoff of $A$ is also negative. Thus choosing $S$ is better for agent $A$, and also for agent $C$ by the symmetric structure of the game. Agent $B$ knows this, so she will also choose $S$. Our multi-period game brings the possibility of indirect

\(^5\)In the rest of this chapter, the belief system is omitted when a PBE is mentioned.
Figure 3.2: Game tree representation of three-person games with a network depicted in Figure 3.1, \( m = 3 \), and \( T = 2 \). In this figure, thresholds \( \theta_A = a, \theta_B = b, \theta_C = c \) is shortened as \( abc \). 
N denotes the move of nature. The decision nodes that are connected by dotted lines are in the same information set. For all decision nodes with two actions, the path on the left is action \( S \) and that on the right is action \( R \). Final payoffs are omitted from the picture.
communication by acting, such like the choice of agent $B$ in period 1, which in turn makes full coordination realizable.

This strategy profile is also the only PBE with full coordination at the end of the game. We have seen that it is not able to choose $(R, R, R)$ simultaneously, either in period 1, or in period 2 after $(S, S, S)$ in period 1. Therefore someone has to be an initiator. Agent $A$ cannot be an initiator because the possibility that $\theta_C = 4$, and the same applies to agent $C$, which means agent $B$ must choose $R$ in period 1 and agents $A$ and $C$ follow in period 2. The above strategy is the only one that can achieve this coordination on $R$.

Now let us consider a slightly modified version of the previous example by decreasing the threshold of an agent, which, at first sight, should increase the likelihood of coordination on choosing $R$. However, as we will see, it has the opposite effect that no PBE with full coordination exists.

**Game 3.2.** Three agents are connected in a line as depicted in Figure 3.1. The thresholds are $\theta_A = 3, \theta_B = 2, \theta_C = 3$, and the minimum threshold is assumed to be $m = 2$. The number of periods is $T = 2$.

We first start with a strategy profile which can achieve full coordination, and show that it is not a PBE. Later on we modify this strategy profile to make it an equilibrium, which then loses the ability to achieve full coordination. Since agents $A$ and $C$ cannot be an initiator, the only way of achieving full coordination is again that $B$ chooses $R$ in period 1 and the other two follow in period 2.

- **Strategy of agent $A$:**

  **Period 1** Choose $R$ if $\theta_A = \theta_B = 2$, otherwise choose $S$.

  **Period 2** Choose $R$ if she has chosen $R$ in period 1, or if agent $B$ has chosen $R$ in period 1 and $\theta_A \leq 3$, otherwise choose $S$.

- **Strategy of agent $B$:**

  **Period 1** Choose $R$ if $\langle \theta_B = 2 \land \langle \theta_A \neq 4 \lor \theta_C \neq 4 \rangle \rangle$, or if $\langle \theta_B = 3 \land \theta_A \neq 4 \land \theta_C \neq 4 \rangle$, otherwise choose $S$.

  **Period 2** Choose $R$ if she has chosen $R$ in period 1, otherwise choose $S$.

- **Strategy of agent $C$:**

  **Period 1** Choose $R$ if $\theta_C = \theta_B = 2$, otherwise choose $S$.

  **Period 2** Choose $R$ if she has chosen $R$ in period 1, or if agent $B$ has chosen $R$ in period 1 and $\theta_C \leq 3$, otherwise choose $S$. 
This strategy profile is not a PBE, since agent \( A \) (and \( C \)) has an incentive to deviate her strategy in period 2. This is because when \( \theta_B = 2 \), initiation behavior of \( B \) can be a tricking behavior since agent \( B \) only needs one additional agent to follow her in order to obtain positive payoff. This means even if \( \theta_C = 4 \), agent \( B \) will benefit from \( A \)'s following, whereas agent \( A \) will end up with a negative payoff. Thus, it is better for agent \( A \) (and \( C \)) to choose another strategy described below.

- **Strategy of agent \( A \)** (modified):

  **Period 1** Choose \( R \) if \( \theta_A = \theta_B = 2 \), otherwise choose \( S \).

  **Period 2** Choose \( R \) if she has chosen \( R \) in period 1, or if agent \( B \) has chosen \( R \) in period 1 and \( \langle \theta_A \leq 3 \land \theta_B = 3 \rangle \), otherwise choose \( S \).

Indeed, this modified strategy of agent \( A \) (and \( C \)) is the unique best response in order to avoid the tricking behavior of \( B \), since the initiation of \( B \) can be a tricking behavior only if \( \theta_B = 2 \). Other strategies that can also avoid the tricking behavior of \( B \) may exist, but they are not best response to strategies of \( B \) that make \( B \) choose \( R \) in period 1.

After the modification of \( A \) (and \( C \)), agent \( B \) now has to modify her strategy in period 1 since no one will follow her initiation in period 2, making her payoff negative. Now we have the next strategy of agent \( B \).

- **Strategy of agent \( B \)** (modified):

  **Period 1** Choose \( R \) if \( \langle \theta_B = 2 \land \langle \theta_A = 2 \lor \theta_C = 2 \rangle \rangle \) or \( \langle \theta_B = 3 \land \theta_A \neq 4 \land \theta_C \neq 4 \rangle \), otherwise choose \( S \).

  **Period 2** Choose \( R \) if she has chosen \( R \) in period 1, otherwise choose \( S \).

The modified strategy of \( B \) together with the modified strategies of \( A \) and \( C \) constitute a PBE strategy profile. However, this profile leads to a full coordination if and only if the thresholds are either \( \theta_A = \theta_B = \theta_C = 2 \) or \( \theta_A = \theta_B = \theta_C = 3 \). In Game 3.2 with \( \theta_A = 3, \theta_B = 2, \theta_C = 3 \), agent \( B \) cannot take the first move because the message behind her action is not credible to the other agents. Indeed, the best response of \( B \) to the modified strategy of \( A \) (and \( C \)) is to stay with \( S \) in both periods 1 and 2. If \( B \) chooses \( S \) in period 1, the modified strategy of agent \( A \) (and \( C \)) is still a best response. Therefore, in any PBE strategy profile there is no full coordination in Game 3.2.

Comparing Game 3.2 with Game 3.1, it is interesting that although the threshold of some agents (here agent \( B \)) is lowered, full coordination becomes even harder to be achieved. In contrast, in the single-period version of the game, lower thresholds generally makes it easier to coordinate.
Figure 3.3: A line network with three agents and their thresholds: $\theta_A = 2, \theta_B = 2, \theta_C = 3$.

Another interesting observation is that restriction of information flow may help with coordination, which again is not a property of the single-period version game. Let us look at the following two games.

**Game 3.3.** Three agents are connected in a line as depicted in Figure 3.1. The thresholds are $\theta_A = 2, \theta_B = 2, \theta_C = 3$, and the minimum threshold is assumed to be $m = 2$. The number of periods is $T = 2$.

**Game 3.4.** Three agents connected in a line as depicted in Figure 3.3. Here agents A and B can observe each other, agent C can observe agent B, but agent B cannot observe agent C. The thresholds are $\theta_A = 2, \theta_B = 2, \theta_C = 3$, and the minimum threshold is assumed to be $m = 2$. The number of periods is $T = 2$.

Game 3.3 and 3.4 differ only in the observation structure. In Game 3.3, the strategy profile that combines the modified strategy of agent A (and C) and the modified strategy of agent B in Game 3.2 is still a PBE, which implies the outcome that agents A and B choose R in period 1 but agent C does not follow. This is because the message from the action of B is not credible to C, i.e., it could be a tricking behavior. However, in Game 3.4, the initiation of B cannot be interpreted as a tricking behavior since C can observe B but B cannot observe C. Therefore agent B does not take into account the choice of agent C. This implies that B’s action can be taken as a reliable signal about the coordination capability of the neighborhood of B, here of neighbor A. In this sense Game 3.4 is equivalent to that A and B play a two-person coordination game and C moves after observing the action of B. The following strategy profile for Game 3.4 corresponds to the modified strategy profile above.

- **Strategy of agent A in Game 3.4:**
  - **Period 1** Choose $R$ if $\theta_A = \theta_B = 2$, otherwise choose $S$.
  - **Period 2** Choose $R$ if she has chosen $R$ in period 1, otherwise choose $S$.

- **Strategy of agent B in Game 3.4:**
  - **Period 1** Choose $R$ if $\theta_A = \theta_B = 2$, otherwise choose $S$. 
**Period 2** Choose $R$ if she has chosen $R$ in period 1, otherwise choose $S$.

- Strategy of agent $C$ in Game 3.4:

  **Period 1** Choose $S$.

  **Period 2** Choose $R$ if $B$ has chosen $R$ in period 1 and $\theta_C \leq 3$, otherwise choose $S$.

This strategy profile is a PBE. It allows $A$ and $B$ to choose $R$ in period 1 and $C$ follows in period 2. From Game 3.3 and Game 3.4 we can see, restriction in information flow (removing the link from $C$ to $B$) plays a positive role in achieving full coordination.

### 3.3.2 Some four-person games

So far we observed that decreasing the threshold of some agents can destroy coordination, and restricting observability can enable coordination, in some three-person games. Here we continue to discuss games with four agents connected in a square network to show that the above properties are not special for three-person games but more general.

**Game 3.5.** Four agents are connected as depicted in Figure 3.4 with thresholds $\theta_U = \theta_V = \theta_W = \theta_X = 3$. The minimum threshold is $m = 2$ and the number of periods is $T = 3$.

Game 3.5 has the following pure strategy PBE. In this strategy profile all agents play the same strategy, so we only state the strategy under the name of agent $U$.

- Strategy of agent $U$:

  **Period 1** Choose $R$ if $\langle \theta_U = 2 \land \theta_V = 2 \lor \theta_W = 2 \rangle$ or $\langle \theta_U = 3 \land \theta_V \leq 3 \land \theta_W \leq 3 \rangle$, otherwise choose $S$.  

Figure 3.4: A four-person square network.
**Period 2** Choose $R$ if

1. $\theta_U = 2, \exists i \in \{V, W\}$ that $i$ has chosen $R$ in period 1; or
2. $\theta_U = 3, \exists i \in \{V, W\}$ with $\theta_i = 3$ and that $i$ has chosen $R$ in period 1; or
3. $\theta_U = 4, \forall i \in \{V, W\}, \theta_i \leq 3$ and that $i$ has chosen $R$ in period 1.

Otherwise choose the same action as in period 1.

**Period 3** Choose $R$ if $\theta_U = 4, \forall i \in \{V, W\}, \theta_i \leq 3$ and that $i$ has chosen $R$ in period 2, otherwise choose the same action as in period 2.

This strategy guarantees full coordination such that all agents choose action $R$ in the first period. Intuitively, although there are four agents in the network, each agent with her two neighbors form a local situation which is described in Game 3.1. Therefore, regardless of the threshold of the fourth agent, the three agents can achieve full coordination. Since each agent is in the center of a line, she is an initiator. Note that Chwe (1999) comes to a different conclusion for the single-period counterpart of Game 3.5. In his setting, no agent will choose $R$ since though each agent in the network knows that there are at least three agents with threshold three, no one is sure about whether her neighbors also have the same information. In other words, the fact that at least three agents are with threshold three is not a common knowledge. In Game 3.5, however, each agent understands the potential of communicating this fact by the credible signal of choosing $R$ in period 1. This implies that full coordination occurs even before the signal can be interpreted, i.e., the mere potential of sending this signal is enough to stimulate full coordination in the first period.

Using some four-person games with the same observation structure, we can see that with the above strategy profile it becomes possible to achieve full coordination when some thresholds are increased. Consider the three games depicted in Figures 3.5a – 3.5c. All these games are variations of Game 3.5. The above strategy profile can also be used in these three games without modification, hence it is a PBE of each of the three games.

In the game in Figure 3.5a, the above strategy profile cannot achieve full coordination. Agents $V$ and $X$ choose action $R$ in period 1, but agents $U$ and $W$ do not follow in the succeeding periods. This is because the initiation of the neighbor with threshold 2 can be a *tricking* behavior. In the game in Figure 3.5b, the threshold $\theta_X$ has been raised from 2 to 3. In this case, agent $X$ initiates action $R$ in period 1, agents $V$ and $W$ follow in period 2, and agent $U$ follows in period 3, which leads to a full coordination. Note that agent $V$ cannot be an initiator since it can be interpreted as a *tricking* behavior by agents $X$ and $U$. If we further raise the threshold $\theta_U$ from 2 to 4 as shown in Figure 3.5c, the outcome of the above strategy profile is that all agents choose $S$ in all periods. The difficulty of achieving full coordination on $R$ under this strategy profile is not monotonic in the values of thresholds.
In the game represented by Figure 3.5d, the link from $W$ to $X$ is deleted compared to Figure 3.5a. For the game of Figure 3.5d we consider the strategy profile in which agents $U$ and $V$ take the above strategy and agents $X$ and $W$ take the strategies described as follows.

- Strategy of agent $X$ in the game of Figure 3.5d:
  
  **Period 1** Choose $R$ if $\theta_X = \theta_V = 2$, otherwise choose $S$.
  
  **Period 2** Choose $R$ if $\theta_X = \theta_V = 3$ and $V$ has chosen $R$ in period 1, otherwise choose the same action as in period 1.
  
  **Period 3** Choose the same action as in period 2.

- Strategy of agent $W$ in the game of Figure 3.5d:
  
  **Period 1** Choose $R$ if $\theta_W = \theta_U = 2$, otherwise choose $S$.
  
  **Period 2** Choose $R$ if
  
  $\theta_W = 3$, $\theta_X = 2$ and $X$ has chosen $R$ in period 1, or
  
  $\theta_W = 3$, $\theta_U = 3$ and $U$ has chosen $R$ in period 1, or
  
  $\theta_W = 4$, $\theta_U \leq 3$ or $\theta_X = 2$) and both $U$ and $X$ have chosen $R$ in period 1.
Otherwise choose the same action as in period 1.

**Period 3** Choose $R$ if $\theta_W = 4$, $\forall i \in \{U, X\}$, $\theta_i \leq 3$ and that $i$ has chosen $R$ in period 2, otherwise choose the same action as in period 2.

It is not difficult to verify that this strategy profile is a PBE in the game depicted in Figure 3.5d. It gives an outcome that agents $V$ and $X$ choose $R$ in period 1, agent $W$ follows in period 2, and finally agent $U$ follows in period 3, leading to a full coordination in the end. Since agent $X$ can not observe agent $W$, the action of $X$ is truthful to $W$, which implies that $W$ can safely follow the initiation of $X$. This is another example showing that fewer connections could help coordination.

In the last part of this section, we restrict our attention to games with network structures that every pair of connected agents is mutually observable. In Table 3.1, all non-isomorphic four nodes connected networks with mutually observable neighbors are shown. Examples of thresholds at which no full coordination can be achieved are shown for each network except the complete network. However, by increasing the thresholds of some agents, full coordination becomes achievable. For the first four networks this higher threshold vector could be the one with common threshold 3, and for the fifth network, the threshold vector could be the one with common threshold 4. These results suggest that the non-monotonic property of achievability of full coordination to threshold vector is a general phenomenon rather than a phenomenon that only holds in special situations.

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6We say two networks $G$ and $G'$ are isomorphic if there is a one-to-one mapping $\varphi$ from the set of nodes of $G$ to the set of nodes of $G'$ such that if there is a link from $x$ to $y$ in $G$, there is also a link from $\varphi(x)$ to $\varphi(y)$ in $G'$. Further concepts about networks (graphs) can be found in Diestel (2010).
Table 3.1: Non-isomorphic connected networks with 4 nodes, where every pair of connected agents is mutually observable.

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<thead>
<tr>
<th>Network</th>
<th>Example of threshold configuration lacking potential of full coordination</th>
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<tbody>
<tr>
<td><img src="image" alt="Network 1" /></td>
<td><img src="image" alt="Example 1" /></td>
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<tr>
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<td><img src="image" alt="Network 3" /></td>
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<tr>
<td><img src="image" alt="Network 4" /></td>
<td><img src="image" alt="Example 4" /></td>
</tr>
<tr>
<td><img src="image" alt="Network 5" /></td>
<td><img src="image" alt="Example 5" /></td>
</tr>
<tr>
<td><img src="image" alt="Network 6" /></td>
<td>None, unless some agents have threshold 5.</td>
</tr>
</tbody>
</table>

Note: For the first five networks, it is possible to achieve full coordination by increasing the thresholds of some agents. For the first four networks this is the case for each agent being with threshold 3, and for the fifth network with threshold 4, respectively.
3.3.3 \textit{n-person games on a line network}

Here we consider an \textit{n-person game} on a line network where every pair of connected agents is mutually observable. The minimum threshold is $m = 2$, and the penalty is $P \rightarrow \infty$. The following strategy is a PBE of the games sharing these conditions. For each agent $i$,

- If $\theta_i = 2$
  - Choose $R$ in period 1 if one of her neighbors has threshold 2.
  - Choose $R$ in period $k > 1$ if one of her neighbors, or herself has chosen $R$ in period $k - 1$.

- If $\theta_i = 3$
  - Choose $R$ in period 1 if two neighbors (when there are) have threshold at most 3.
  - Choose $R$ in period $k > 1$ if one of her neighbors has threshold 3, or herself has chosen $R$ in period $k - 1$.

- Otherwise choose $S$.

Obviously this strategy profile covers the PBE of Game 3.2 and 3.3 as a special case. It is also easy to see that when the threshold of a terminal agent (an agent on an end of the line) is 3 and her neighbor has threshold 2, there will be no full coordination in this game, because the terminal agent will neither initiate nor follow her neighbor. For intermediate agents, a threshold of at most 3 is a necessary condition for choosing $R$.

3.4 \textit{A case study of equilibria in behavior strategies}

It has been shown that there exists no pure strategy PBEs with full coordination in Game 3.2 when $P$ is taken sufficiently large. Here, we would like to refine the discussion and to specify the range of $P$ for which a full coordination is possibly achieved. In order to analyze this question in detail we extend the scope to the more general approach of behavior strategies.\footnote{In any finite extensive game with perfect recall, Kuhn (1953) shows that for every behavior strategy there is a mixed strategy that has an equivalent payoff, and vice versa. In our analysis all the games are finite and with perfect recall, therefore we can restrict our attention to behavior strategies since it is easier to represent.}

Interestingly, this will reveal an additional property that the likelihood of coordination can be an increasing function of the penalty $P$ in behavior strategies.

We first discuss the single-period version of Game 3.2 and its Bayesian Nash equilibria (BNEs) and than look at its multi-period counterpart. Due to the symmetric structure of the game we focus on strategy profiles that are symmetric with respect to $A$ and $C$. In order to
keep the analysis tractable, we assume that pure strategy profile \((R, R)\) is played for any pair of mutually connected agents with thresholds \((2,2)\) in the first period.\(^8\) We assume uniform prior probabilities over all possible threshold vectors. That is, in the extensive form game, nature chooses each threshold vector with equal probability. As a result, after nature’s move, the posterior probabilities of unobservable thresholds are also uniform for each agent.

### 3.4.1 Single-period games

Here we consider the single-period version of Game 3.2. The following arguments derive the necessary conditions of its BNEs.

**From the viewpoint of A** Agent A cannot distinguish between thresholds \(\theta_C = 2, \theta_C = 3\) and \(\theta_C = 4\), thus she must consider all these cases. For \(\theta_C = 2\), agents B and C will choose action \(R\). For \(\theta_C = 3\), suppose that agent B and C chooses \(R\) with probability \(q\) and \(Q\) respectively. In this case agents A and C are symmetric, so in equilibrium agent A will also choose action \(R\) with probability \(Q\). In the case \(\theta_C = 4\), agent A will always receive \(-P\) by choosing action \(R\) regardless the choice of B, since C will always choose action S. Because of the uniform prior beliefs assumption, A’s posterior probabilities of these three threshold vectors are equal. Then, the expected payoff of agent A given she chooses \(R\) is

\[
EP_A(R) = \frac{1}{3} + \frac{1}{3} \left\{ qQ + (1-q)(-P) \right\} + \frac{1}{3}(-P) \\
= \frac{1}{3}(1-2P) + \frac{1}{3}(1+P)qQ.
\]

The expected payoff of A given she stays with the status quo is always 0. So if \(EP_A(R) > 0\), choosing \(R\) (or equivalently \(Q = 1\)) is the best reply for A. If \(EP_A(R) < 0\), A should choose \(S\) which means \(Q = 0\). If \(EP_A(R) = 0\), it is indifferent to choose any \(Q \in [0, 1]\). The following formulas summarize the necessary conditions of the values of \(EP_A(R)\) and the consistent values of \(Q\).

\[
\begin{align*}
EP_A(R) > 0 \quad (Q = 1) \quad & \Rightarrow \quad q > \frac{2P-1}{1+P} \quad \Leftrightarrow \quad P < \frac{1-q}{2-q}, \\
EP_A(R) < 0 \quad (Q = 0) \quad & \Rightarrow \quad P > \frac{1}{2}, \\
EP_A(R) = 0 \quad (Q \in [0, 1]) \quad & \Rightarrow \quad \left\{ q \neq 0, Q = \frac{2P-1}{q(1+P)} \right\} \text{ or } \left\{ q = 0, P = \frac{1}{2} \right\}.
\end{align*}
\]

\(^8\)Note that, in the 2-person single-period game with thresholds \((2,2)\) where both agents can observe each other, all NEs in mixed strategies are \((0,0), (1,1), \) and \((P/(1+P), P/(1+P))\), which are vectors of the probabilities of choosing action \(R\) for the two agents.
From the viewpoint of $B$ Each information set of agent $B$ contains a single threshold vector. Let the probability of choosing action $R$ of agents $A$ and $C$ be $\pi$ in equilibrium. For thresholds $\theta_A = \theta_C = 3$, the expected payoff of agent $B$ given she chooses $R$ is

$$EP_B(R) = \left\{ 1 - (1 - \pi)^2 \right\} + (1 - \pi)^2(-P) = 1 - (1 + P)(1 - \pi)^2.$$ 

If $EP_B(R) > 0$ agent $B$ chooses action $R$, if $EP_B(R) < 0$ she chooses $S$. If $EP_B(R) = 0$ she is indifferent in having the probability of choosing $R$ within the closed interval $[0, 1]$. The necessary and sufficient conditions of values of $EP_B(R)$ is summarized below:

$$\begin{align*}
EP_B(R) > 0 & \iff \pi = 1 \text{ or } \pi > 1 - \frac{1}{\sqrt{1 + P}} \iff P < \frac{1}{(1 - \pi)^2} - 1 \\
EP_B(R) < 0 & \iff \pi = 0 \text{ or } \pi < 1 - \frac{1}{\sqrt{1 + P}} \iff P > \frac{1}{(1 - \pi)^2} - 1 \\
EP_B(R) = 0 & \iff P = \frac{1}{(1 - \pi)^2} - 1
\end{align*}$$

BNEs By letting $Q = \pi$, one can derive BNEs in behavior strategies. All pure and behavior strategy BNEs of the single-period version of Game 3.2 and their necessary conditions are summarized in Table 3.2 and illustrated in Figure 3.6. It is not surprising that coexistence of multiple equilibria is possible. Of interest is that there exists regions of small $P$ where the probability of choosing action $R$ for each agent is an increasing function of $P$ (the red and blue curves on $0.5 \leq P \leq 2$ in Figure 3.6). If $P$ is taken sufficiently large (here for $P > 2$), the only equilibrium is that everyone plays $S$ with probability one.

Table 3.2: Behavior BNEs on the equilibrium path of the single-period version of Game 3.2 and their necessary conditions. Strategies are in terms of probability of choosing action $R$.

<table>
<thead>
<tr>
<th>Behavior Strategy Profile</th>
<th>Necessary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1, 1)$</td>
<td>$P \leq 2$</td>
</tr>
<tr>
<td>$(\pi, 1, \pi)$</td>
<td>$\frac{5 - \sqrt{13}}{2} \leq P \leq 2$, $\pi = \frac{2p - 1}{1 + p}$</td>
</tr>
<tr>
<td>$(\pi, q, \pi)$</td>
<td>$\frac{1}{2} \leq P \leq \frac{5 - \sqrt{13}}{2}$, $\pi = 1 - \frac{1}{\sqrt{1 + P}}$, $q = \frac{2p - 1}{1 + p - \sqrt{1 + P}}$</td>
</tr>
<tr>
<td>$(\pi, 0, \pi)$</td>
<td>$P = \frac{1}{2}$, $0 \leq \pi \leq 1 - \frac{\sqrt{3}}{3}$</td>
</tr>
<tr>
<td>$(0, 0, 0)$</td>
<td>$P \geq \frac{1}{2}$</td>
</tr>
</tbody>
</table>
Figure 3.6: Conditions of behavior strategy BNEs on the equilibrium path of the single-period version of Game 3.2 expressed on \((P, \pi)\)-plane. Strategies are in terms of probability of choosing action \(R\).

3.4.2 Multi-period games

There may exist PBEs in behavior strategies in the multi-period version of Game 3.2. Due to her lower threshold and informational advantage of the central position we will focus on coordination initiated by agent \(B\). In particular, we focus on the strategies where \(B\) plays \(R\) in period 1 and \(A\) and \(C\) follow in period 2 with non-negative probabilities. In detail, agent \(B\) starts with \(R\) in period 1 whenever she has threshold 2 and at least one neighbor has threshold less than 4. Agents \(A\) and \(C\), on the other hand, never choose \(R\) in period 1 whenever they have threshold 3 or higher.

As we will show below, in period 2, both agents \(A\) and \(C\) play \(R\) with probability 1 is an equilibrium under the condition \(P < 2\). Another equilibrium is they play \(R\) with probability \((2P - 1)/(1 + P)\) for \((5 - \sqrt{13})/2 \leq P < 2\).

**From the viewpoint of \(A\)** Let \(Q'\) be the probability of agent \(C\) choosing \(R\) in period 2 if her threshold is 3. The expected payoff of \(A\) when she chooses \(R\) in period 2 is

\[
\text{EP}_A(R) = \frac{1}{3} + \frac{1}{3} \left( Q' + (1 - Q')(\!-P\!) \right) + \frac{1}{3} (\!-P\!),
\]
where 1/3's are the posterior probabilities of the threshold $\theta_C = 2$, $\theta_C = 3$ and $\theta_C = 4$. We then have

$$
\begin{align*}
\text{EP}_A(R) > 0 & \quad (Q' = 1) \quad \Rightarrow \quad P < 2, \\
\text{EP}_A(R) < 0 & \quad (Q' = 0) \quad \Rightarrow \quad P > \frac{1}{2}, \\
\text{EP}_A(R) = 0 & \quad (Q' \in [0, 1]) \quad \Rightarrow \quad Q' = \frac{2P - 1}{P + 1}.
\end{align*}
$$

**From the viewpoint of B**: Let $\pi'$ denote the probability of playing $R$ of $A$ and $C$ in the second period. The expected payoff of agent $B$ given she chooses $R$ in period 1 (and thus also in period 2) is

$$
\text{EP}_B(R) = \left(1 - (1 - \pi')^2\right) + (1 - \pi')^2(-P) = 1 - (1 + P)(1 - \pi')^2,
$$

which is similar to the formula in the single-period version of this game. So the necessary and sufficient conditions are similar as stated before.

**PBEs**: By letting $Q' = \pi'$, one can find two PBE strategy profiles under the consideration that $B$ chooses $R$ in the first period. One is that agents $A$ and $C$ choose $R$ for sure in period 2 under the condition $P \leq 2$, and the other is that they choose $R$ with probability $(2P - 1)/(1 + P)$ for $(5 - \sqrt{13})/2 \leq P < 2$. This probability is again an increasing function of $P$. For $P > 2$ the only PBE is that all agents choose $S$ in both period 1 and 2.

### 3.5 Concluding remarks

In this chapter we discussed the role of action as a signaling device in multi-period coordination games when the communication structure is given by a social network. Our discussion identified three interesting phenomena of coordination. First, increasing thresholds can foster coordination as it increases the credibility of the signal of choosing risky behavior. Second, restricting communication can facilitate coordination as it takes away the possibility of cheating. Third, increasing the penalty can help coordination in behavior strategy. Here, the intuition is that in order to counterbalance higher penalty a higher probability of choosing $R$ is needed in equilibrium.

We demonstrated these phenomena by focusing on existence and structural properties of equilibria in small games. Further analysis may be done for games with general communication structures and threshold vectors. However, larger games soon become intractable due to rapid growing of possible network configurations, type space and possible strategies.
with larger number of periods. Different aspects might lead to some interesting new questions. One direction is along the line of the characterization of "minimal sufficient network" in Chwe (2000). In his article, he defines this as the minimal endowment of links in order to achieve coordination. This concept relies on the fact that in his setting of single-period games the existence of equilibria with full coordination is monotonic with respect of presence of links, which is shown to be failed in multi-period games. Therefore, some modifications are needed in order to go further on this direction.

Another possible line of future research is to extend communication possibilities as, e.g., spreading more information such as agents to "tell" their neighbors about observations up to the current period with a positive probability. Last but not least one might wonder about the likelihood of collective action when the network is not static but changes due to, say, strategic aspects of finding like-minded people.