5 | Conformal measure ensembles for percolation and FK-Ising

This chapter is based on [20] with Federico Camia and Demeter Kiss.

Under some general assumptions we construct the scaling limit of open clusters and their associated counting measures in a class of two-dimensional percolation models. Our results apply, in particular, to critical Bernoulli site percolation on the triangular lattice. We also provide conditional results for the critical FK-Ising model on the square lattice. Fundamental properties of the scaling limit, such as conformal covariance, are explored. Applications such as the scaling limit of the largest cluster in a bounded domain and a geometric representation of the magnetization field for the critical Ising model are presented.

5.1 Introduction

Several important models of statistical mechanics, such as percolation and the Ising and Potts models, can be described in terms of clusters. In the last fifteen years, there has been tremendous progress in the study of the geometric properties of such models in the scaling limit. Much of that work has focused on interfaces, that is, cluster boundaries, taking advantage of the introduction of the Schramm-Loewner Evolution (SLE) by Oded Schramm in [71]. In this chapter, we are concerned with the scaling limit of the clusters themselves and their “areas.” More precisely, we analyze the scaling limit of the collection of clusters and the associated counting measures (rescaled by an appropriate power of the lattice spacing).

Our main results are valid under some general assumptions, which can be verified for Bernoulli site percolation on the triangular lattice. Most of the assumptions can be verified also for the FK-Ising model (FK percolation with $q = 2$), but in that case our results are conditional, since we need to assume that the critical FK-Ising percolation model has a unique, conformally invariant, full scaling limit in terms of loops. (The analogous result for Bernoulli percolation was proved in [22]). Such a scaling limit is conjectured to exist and to be described by the Conformal Loop Ensemble (CLE) with parameter $16/3$. Recent progress in that direction has been reported in [29], [51].

Roughly speaking, our main results say that, under suitable assumptions, in a general two-dimensional percolation model, the collection of clusters and their associated counting measures, once appropriately rescaled, has a unique weak limit, in
an appropriate topology. The collection of clusters converges to a collection of closed sets (the “continuum clusters”), while the collection of rescaled counting measures converges to a collection of continuum measures whose supports are the continuum clusters.

Our results are nontrivial at the critical point of the percolation model. For instance, in the case of critical site percolation on the triangular lattice, where a scaling limit in terms of cluster boundaries is known to exist and to be conformally invariant [22] (it can be described in terms of SLE$_6$ curves), we show that the continuum clusters are also conformally invariant, and that the associated measures are conformally covariant. The conformal covariance property of the collection of measures is a consequence of the conformal invariance of the critical scaling limit. Because of this property, we call the collection of measures arising in the scaling limit of a critical percolation model a Conformal Measure Ensemble, as proposed by Federico Camia and Charles M. Newman (see [24] and [19]). In the case of Bernoulli percolation, we also use our results to obtain the scaling limit of the largest clusters in a bounded domain.

The scaling limit of the rescaled counting measures is in the spirit of [38], and indeed we rely heavily on techniques and results from that paper. There is however a significant difference in that we distinguish between different clusters. In other words, we don’t obtain a single measure that gives the combined size of all clusters inside a domain, but rather obtain a collection of measures, one for each cluster. This is the main technical difficulty in this chapter.

When applied to FK percolation, our results have an interesting application to the Ising model. Consider a critical Ising model on the scaled lattice $\eta\mathbb{Z}^2$. Using the FK representation, one can write the total magnetization in a domain $D$ as $\sum_{i} \sigma_i \nu_i^\eta(D)$, where the $\sigma_i$’s are $(\pm 1)$-valued, symmetric random variables independent of each other and everything else, and $\nu_i^\eta = \sum_{u \in C_i} \delta_u$ is the counting measure associated to the $i$-th cluster ($\delta_u$ denotes the Dirac measure concentrated at $u$ and the order of the clusters is irrelevant) and $\nu_i^\eta(D) = |C_i \cap D|$, where $C_i$ is the $i$-th cluster. Camia and Newman [24] noticed that the power of $\eta$ by which one should rescale the magnetization to obtain a limit, as $\eta \to 0$, is the same as the power that should ensure the existence of a limit for the rescaled counting measures. They then predicted that one should be able to give a meaning to the expression “$\Phi^\infty = \sum_{i} \sigma_i \mu_i^0$”, where $\Phi^\infty$ is the limiting magnetization field, obtained from the scaling limit of the renormalized lattice magnetization, and $\{\mu_i^0\}$ is the collection of measures obtained from the scaling limit of the collection of rescaled versions of the counting measures $\{\nu_i^\eta\}$. The existence and uniqueness of the limiting magnetization field was proved in [21], here we complete the program put forward in [24] for the two-dimensional critical Ising model by showing that the Ising magnetization field can indeed be expressed in terms of cluster measures, thus providing a geometric representation (a sort of continuum FK representation based on continuum clusters) for the limiting magnetization field.

### 5.1.1 Definitions and main results

Let $\mathbb{L}$ denote a regular lattice with vertex set $V(\mathbb{L})$ and edge set $E(\mathbb{L})$. For $u$ and $v$ in $V(\mathbb{L})$, we write $u \sim v$ if $(u,v) \in E(\mathbb{L})$. We are interested in Bernoulli percolation and FK-Ising percolation in $\mathbb{L}$ with parameter $p$. When we talk about FK-Ising
percolation, $\mathbb{L}$ will be the square lattice $\mathbb{Z}^2$. The FK clusters are defined as illustrated in Figure 5.1, and we think of them as closed sets whose boundaries are the loops in the medial lattice shown in Figure 5.1 (see [42] for an introduction to FK percolation).

When dealing with Bernoulli percolation, $\mathbb{L}$ will be the triangular lattice $\mathbb{T}$, with vertex set

$$V(\mathbb{T}) := \{x + y\epsilon \in \mathbb{C} \mid x, y \in \mathbb{Z}\},$$

where $\epsilon = e^{\pi i/3}$. The edge set $E(\mathbb{T})$ of $\mathbb{T}$ consists of the pairs $u, v \in V$ for which $\|u - v\|_2 = 1$. Further, let $H_u$ denote the regular hexagon centered at $u \in V(\mathbb{T})$ with side length $1/\sqrt{3}$ with two of its sides parallel to the imaginary axis. Clusters are maximal connected components of open or closed hexagons (see [41] for an introduction to Bernoulli percolation).

Let $\eta > 0$ and consider Bernoulli percolation on $\eta \mathbb{T}$ or the FK-Ising model on $\eta \mathbb{Z}^2$. We think of open and closed clusters as compact sets. To distinguish between them, we will call open clusters `red' and closed clusters `blue' (we deviate from the usual terminology of open and closed clusters on purpose: we reserve the words `open' and `closed' to describe the topological properties of sets). Let $\sigma_\eta$ denote the union of the red clusters in $\eta \mathbb{L}$.

Further, let

$$\Lambda_r := \{z \in \mathbb{C} \mid |\mathfrak{R}z| \leq r, |\mathfrak{I}z| \leq r\}$$

denote the ball of radius $r$ around the origin in the $L^\infty$ norm. We set $\Lambda_r(u) = u + \Lambda_r$.

Our aim is to understand the limit of the set $\sigma_\eta$ as $\eta$ tends to 0. It is easy to see that the limit of $\sigma_\eta$ in the Hausdorff topology as $\eta \to 0$ is trivial: it is the empty set when $p = 0$ and a.s. $\mathbb{C}$ for $p > 0$. Hence we concentrate on the connected components, i.e. clusters, of $\sigma_\eta$ with diameter at least $\delta$ for some fixed $\delta > 0$. It is well-known (see
for instance [4]) that, again, we get trivial limits unless \( p = p_c \). (For \( p < p_c \) the limit of each of the clusters is the empty set, while for \( p > p_c \) the limit of the unique largest clusters is dense in \( \mathbb{C} \), with the other clusters having the empty set as a limit.) Hence we consider \( p = p_c \) in the following, and state informal versions of our main results after some additional definitions. The precise versions of our results are postponed to later sections.

For a set \( A \subseteq \mathbb{C} \) and \( u, v \in \mathbb{C} \) we write \( u \leftrightarrow_A v \) if there is a red path running in \( A \) which connects \( u \) to \( v \). When \( A \) is omitted, it is assumed to be \( \mathbb{C} \). Let \( \text{diam}(A) \) denote the \( L^\infty \) diameter of \( A \). For \( u \in \eta V \) denote by \( C^n(u) \) the connected component (i.e. cluster) of \( u \) in \( \sigma_{\eta} \). For \( D \) a simply connected domain with piece-wise smooth boundary, let \( C^n_D(\delta) \) denote the collection of connected components of \( \sigma_{\eta} \), which are contained in \( D \) and have diameter larger than \( \delta \). That is,

\[
C^n_D(\delta) := \{ C^n(u) \mid u \in \eta V, C^n(u) \subseteq D, \text{diam}(C^n(u)) \geq \delta \}.
\]  

(5.1)

On many places \( D \) is taken to be \( \Lambda_k \), in that case we simplify notation by writing \( C^n_k(\delta) := C^n_{\Lambda_k}(\delta) \). Finally let

\[
C^n(\delta) = \bigcup_{k \in \mathbb{N}} C^n_k(\delta)
\]  

(5.2)

denote the collection of all connected components of \( \sigma_{\eta} \) with diameter at least \( \delta \).

In the following theorem, distances between subsets of \( \mathbb{C} \) will be measured by the Hausdorff distance built on the \( L^\infty \) distance in \( \mathbb{C} \): For \( A, B \subseteq \mathbb{C} \),

\[
d_H(A, B) := \inf \{ \varepsilon > 0 \mid A + \Lambda_\varepsilon \supseteq B \text{ and } B + \Lambda_\varepsilon \supseteq A \},
\]  

(5.3)

where \( A + \Lambda_\varepsilon := \{ x + y \in \mathbb{C} : x \in A, y \in \Lambda_\varepsilon \} \).

Let \( \hat{\mathbb{C}} \) be the one-point (Alexandroff) compactification of \( \mathbb{C} \), i.e. the Riemann sphere \( \hat{\mathbb{C}} := \mathbb{C} \cup \{ \infty \} \). A distance between subsets of \( \hat{\mathbb{C}} \) which is equivalent to \( d_H \) on bounded sets is defined via the metric on \( \mathbb{C} \) with distance function

\[
\Delta(u, v) := \inf_{\varphi} \int_0^1 \frac{1}{1 + |\varphi'(s)|^2} ds,
\]

where we take the infimum over all curves \( \varphi(s) \) in \( \mathbb{C} \) from \( u \) to \( v \) and \( | \cdot | \) denotes the Euclidean norm.

The distance \( D_H \) between sets is then defined by

\[
D_H(A, B) := \inf\{ \varepsilon > 0 \mid \forall u \in A : \exists v \in B : \Delta(u, v) \leq \varepsilon \text{ and vice versa} \}.
\]  

(5.4)

The distance between finite collections i.e., sets of subsets of \( \mathbb{C} \), denoted by \( \mathcal{I}, \mathcal{I}' \), is defined as

\[
\min_{\phi} \max_{S \in \mathcal{I}} d_H(S, \phi(S))
\]  

(5.5)

where the infimum is taken over all bijections \( \phi : \mathcal{I} \to \mathcal{I}' \). In case \( |\mathcal{I}| \neq |\mathcal{I}'| \) we define the distance to be infinite. To account for possibly infinite collections, \( \mathcal{I} \) and \( \mathcal{I}' \), of subsets of \( \hat{\mathbb{C}} \), we define

\[
dist(\mathcal{I}, \mathcal{I}') := \inf\{ \varepsilon > 0 \mid \forall A \in \mathcal{I} \exists B \in \mathcal{I}' : D_H(A, B) \leq \varepsilon \text{ and vice versa} \}.
\]  

(5.6)

Our first result is the following, see Theorem 5.5.1 for a slightly stronger version.
Theorem 5.1.1. Let $k > \delta > 0$. Then $\mathcal{C}^{\eta}_{k}(\delta)$ converges in distribution, in the topology (5.5), to a collection of closed sets which we denote by $\mathcal{C}^{0}(\delta)$. Moreover, as $\delta \to 0$, $\mathcal{C}^{\eta}_{k}(\delta)$ has a limit in the metric (5.6), which we denote by $\mathcal{C}^{0}_{k}$.

The next natural question to ask is whether we can extract some more information from the scaling limit. In particular, can we count the number of vertices in each of the clusters in $\mathcal{C}^{\eta}(\delta)$ in the limit as $\eta$ tends to 0? As we will see below, the number of vertices in the large clusters goes to infinity, hence we have to scale this number to get a non-trivial result. The correct factor is $\eta^{-2}\pi^{\eta}_{1}(\eta,1)$, where $\pi^{\eta}_{1}(\eta,1)$ denotes the probability that 0 is connected to $\partial\Lambda_{1}$ in $\sigma_{\eta}$. We arrive to the informal formulation of our next main result after some more notation.

For $S \subset \mathbb{C}$ let $\mu^{\eta}_{S}$ denote the normalized counting measure of its vertices, that is,

$$
\mu^{\eta}_{S} := \frac{\eta^{2}}{\pi^{\eta}_{1}(\eta,1)} \sum_{u \in S \cap \eta V} \delta_{u},
$$

(5.7)

where $\delta_{u}$ denotes the Dirac measure concentrated at $u$. Further, let $\mathcal{M}^{\eta}_{k}(\delta)$ denote the collection of normalized counting measures of the clusters in $\mathcal{C}^{\eta}_{k}(\delta)$. That is,

$$
\mathcal{M}^{\eta}_{k}(\delta) := \{ \mu^{\eta}_{C} | C \in \mathcal{C}^{\eta}_{k}(\delta) \}. \tag{5.8}
$$

Similarly $\mathcal{M}^{\eta}(\delta) := \{ \mu^{\eta}_{C} | C \in \mathcal{C}^{\eta}(\delta) \}$. We use the Prokhorov distance for the normalized counting measures. For finite Borel measures $\mu, \nu$ on $\mathbb{C}$, it is defined as

$$
d_{P}(\mu, \nu) := \inf \{ \varepsilon > 0 | \mu(S) \leq \nu(S^{\varepsilon}), \nu(S) \leq \mu(S^{\varepsilon}) \text{ for all closed } S \subset \mathbb{C} \},
$$

where $S^{\varepsilon} = S + \Lambda_{\varepsilon}$. Then we construct a metric on collections of Borel measures from $d_{P}$ similarly to (5.5). We also introduce a distance Dist between (infinite) collections of measures which is the same as (5.6) but with collections of sets replaced by collections of measures and with the distance $D_{H}$ replaced by the Prokhorov distance $d_{P}$.

We arrive to the following result. See Theorem 5.7.2 for a slightly stronger version.

Theorem 5.1.2. Let $k > \delta > 0$, then $\mathcal{M}^{\eta}_{k}(\delta)$ converges in distribution to a collection of finite measures which we denote by $\mathcal{M}^{0}_{k}(\delta)$. Moreover, as $\delta \to 0$, $\mathcal{M}^{\eta}_{k}(\delta)$ has a limit in the metric Dist, which we denote by $\mathcal{M}^{0}_{k}$.

The next theorem is a full-plane analogue of Theorems 5.1.1 and 5.1.2.

Theorem 5.1.3. Let $\mathbb{P}_{k}$ denote the joint distribution of $(\mathcal{C}^{0}_{k}, \mathcal{M}^{0}_{k})$. There exists a probability measure $\mathbb{P}$ on the space of collections of subsets of $\mathbb{C}$ and collections of measures, which is the full plane limit of the probability measures $\mathbb{P}_{k}$ in the sense that, for every bounded domain $D$, the restriction $\mathbb{P}_{k}|_{D}$ of $\mathbb{P}_{k}$ to $(\mathcal{C}^{0}_{D}, \mathcal{M}^{0}_{D})$ converges to the restriction $\mathbb{P}|_{D}$ of $\mathbb{P}$ to $(\mathcal{C}^{0}_{D}, \mathcal{M}^{0}_{D})$ as $k \to \infty$.

The next theorem shows that the collections of clusters and measures from the previous theorem are invariant under rotations and translations, and transform covariantly under scale transformations. (The theorem could be extended to include more general fractal linear (Möbius) transformations by restricting to the Riemann sphere minus a neighbourhood of the origin and of infinity. For simplicity, we restrict
attention to linear transformations that map infinity to itself. The random variables
with distribution $\mathbb{P}$ introduced in the previous theorem are denoted by $(\mathcal{E}^0, \mathcal{M}^0)$.

**Theorem 5.1.4.** Let $f$ be a linear map from $\mathbb{C}$ to $\mathbb{C}$, that is $f(z) = rz + t$ with
$r, t \in \mathbb{C}$. Assume that

$$\lim_{\eta \to 0} \pi_1^\eta(a, b) = \left(\frac{a}{b}\right)^{\alpha_1 + o(1)}$$

for all $b > a > \eta$ and some $\alpha_1 \in [0, 1]$, where $o(1)$ is understood as $b/a \to \infty$. We set

$$f(\mathcal{E}^0) := \{f(C) : C \in \mathcal{E}^0\}, \text{ and}$$

$$f(\mathcal{M}^0) := \{\mu^0_* : \mu^0 \in \mathcal{M}^0\}$$

where $\mu^0_*$ is the modification of push-forward measure of $\mu^0$ along $f$ defined as

$$\mu^0_*(B) := |r|^{2-\alpha_1} \mu^0(f^{-1}(B))$$

for Borel sets $B$. Then the pairs $(f(\mathcal{E}^0), f(\mathcal{M}^0))$ and $(\mathcal{E}^0, \mathcal{M}^0)$ have the same distribution.

**Remark 5.1.5.** In the case of Bernoulli percolation, we will prove invariance/covariance
under all conformal maps between any two bounded domains with piecewise smooth boundaries (see Theorems 5.8.6 and 5.8.8).

**Organization of this chapter**

In the next section we discuss some applications of our results. First we consider
applications for Bernoulli percolation on the triangular lattice. Secondly we provide
a geometric representation for the magnetization field of the critical Ising model in
terms of FK clusters.

In Section 5.3 we introduce the main tools and assumptions which we use through-
out this chapter, namely the loop process, the quad-crossing topology, arm events and
the general assumptions under which we prove our main results. We finish Section
5.3 with checking that the assumptions hold for critical Bernoulli percolation on $\mathbb{T}$
and comment on the validity of our assumptions in the critical FK-Ising model. In
Sections 5.4 - 5.7 we give precise versions and proofs of Theorems 5.1.1, 5.1.2 and
5.1.3.

We investigate some fundamental properties of the continuum clusters and their
normalized counting measures in Section 5.8. In particular, we also discuss the con-
formal invariance and covariance properties of the clusters in this section. We finish
this chapter in Section 5.9 where we prove the convergence of the largest clusters for
Bernoulli percolation in a bounded domain.
5.2 Applications

5.2.1 Largest Bernoulli percolation clusters and conformal invariance/covariance

Our first application concerns the scaling limit of the largest percolation clusters in a bounded domain with closed (blue) boundary condition. Denote by $\mathcal{M}^\eta_{(i)}$ the $i$-th largest cluster in $\Lambda_1 \cap \sigma_\eta$, where we measure clusters according to the number of vertices they contain.

In a sequence of papers, the behaviour of the normalized number of vertices,

$$\frac{|\mathcal{M}^\eta_{(i)}|}{\eta^{-2} \pi_1^\eta(\eta, 1)} = \mu^\eta_{\mathcal{M}^\eta_{(i)}}(\Lambda_1),$$

was investigated for $\eta > 0$ and $i \geq 1$. Probably the first such results appeared in [16] and [17]. Using Theorems 5.1.1 and 5.1.2 and results in Section 5.6 about convergence of clusters and portions of clusters in bounded domains, we deduce the following theorem.

**Theorem 5.2.1.** For all $i \in \mathbb{N}$, both the cluster $\mathcal{M}^\eta_{(i)}$ and its normalized counting measure $\mu^\eta_{\mathcal{M}^\eta_{(i)}}$ converges in distribution to a closed set $\mathcal{M}^0_{(i)}$ and a measure $\mu^0_{\mathcal{M}^0_{(i)}}$ as $\eta \to 0$.

Recently some of the results from [16, 17] where sharpened [10, 11, 56]. These sharpened results, in combination with Theorem 5.2.1, imply that the distribution of $\mu^0_{\mathcal{M}^0_{(i)}}(\Lambda_1)$ has no atoms [11], that its support is $(0, \infty)$ [10] and that it has a stretched exponential upper tail [56].

It is a celebrated result of Smirnov [77] that the critical site percolation on the triangular lattice is conformally invariant in the limit as $\eta \to 0$. See also [22]. As we show, under certain technical conditions, that this implies that the collections of large clusters in the limit as $\eta \to 0$ are also conformally invariant, while their normalized counting measures are conformally covariant by the results in [38]. We arrive to the following, which is stated in a slightly stronger form as Theorems 5.8.6 and 5.8.8.

**Theorem 5.2.2.** Let $f$ be conformal map defined on an open neighbourhood of $\Lambda_1$, and $D = f(\Lambda_1)$. We set

$$f(\mathcal{C}^0_{\Lambda_1}) := \{ f(C) : C \in \mathcal{C}^0_{\Lambda_1} \}, \text{ and}$$

$$f(\mathcal{M}^0_{\Lambda_1}) := \{ \mu^{0*} : \mu^0 \in \mathcal{M}^0_{\Lambda_1} \}$$

where $\mu^{0*}$ is the modification of push-forward measure of $\mu^0$ along $f$ defined as

$$\mu^{0*}(B) := \int_{f^{-1}(B)} \left| f'(z) \right|^{91/48} d\mu^0(z)$$

for Borel sets $B$.

Then the pairs $(f(\mathcal{C}^0_{\Lambda_1}), f(\mathcal{M}^0_{\Lambda_1}))$ and $(\mathcal{C}^0_D, \mathcal{M}^0_B)$ have the same distribution.

The proof of Theorem 5.2.1 will be presented in Section 5.9 and the proof of Theorem 5.2.2 in Section 5.8.2.
5.2.2 Geometric representation of the critical Ising magnetization field

In this section we give a geometric representation for the scaling limit of the critical Ising magnetization in two dimensions. The existence and uniqueness of the limiting magnetization field was proved in [21], but already in [24] it was heuristically argued that the Ising magnetization field should be expressible in terms of the limiting cluster measures of the FK-Ising clusters, giving a sort of continuum FK representation based on continuum clusters.

Consider a two-dimensional critical Ising model on \( \eta \mathbb{Z}^2 \) and its FK representation (see, e.g., [42]). We denote by \( \Phi^\infty \) the limiting magnetization field constructed in [21] in the limit \( \eta \to 0 \); it is a random distribution acting on the Sobolev space \( \mathcal{H}^3 \). We also introduce the \( \varepsilon \)-cutoff magnetization \( \Phi^\infty_\varepsilon \), define as

\[
\Phi^\infty_\varepsilon := \sum_{j : \text{diam}(C_j) > \varepsilon} \sigma_j \mu^0_{C_j},
\]

where the sum is over all clusters of diameter larger than \( \varepsilon \) (the order of the sum is irrelevant), the \( \sigma_j \)'s are i.i.d. symmetric \((\pm 1)\)-valued random variables, the \( \mu^0_{C_j} \)'s are the scaling limits of the FK-Ising normalized counting measures, and we think of \( \Phi^\infty_\varepsilon \) as a random measure acting on the space \( C_0^\infty \) of infinitely differentiable functions with bounded support. We will show that the cutoff magnetization \( \Phi^\infty_\varepsilon \) provides a good approximation of the magnetization field \( \Phi^\infty \); since we will only apply \( \Phi^\infty_\varepsilon \) to functions with bounded support, the infinite sum in its definition will reduce to a finite sum, so we don't need to specify an order for the infinite sum.

Under the assumption that the critical FK-Ising percolation model has a unique, conformally invariant, full scaling limit in terms of loops we prove the following theorem (see Section 5.3.3 for a precise formulation of Assumption IV).

**Theorem 5.2.3.** If Assumption IV holds for FK-Ising percolation, then for any \( f \in C_0^\infty \), as \( \varepsilon \to 0 \), \( \langle \Phi^\infty_\varepsilon, f \rangle \) is an \( L^2 \) random variable and moreover it converges to \( \langle \Phi^\infty, f \rangle \) in the \( L^2 \) norm.

**Proof.** As explained in Section 2.2.5 of [21], for any \( f \in C_0^\infty \), \( \langle \Phi^\infty, f \rangle \) can be approximated in the \( L^2 \) norm using functions that are linear combinations of indicator functions of dyadic squares. Therefore, without loss of generality, we can restrict our attention to the magnetization in the unit square: \( \langle \Phi^\infty, 1_{[0,1]^2} \rangle \).

Using the triangle inequality, for any \( \eta > 0 \), we can write

\[
\| \langle \Phi^\infty, 1_{[0,1]^2} \rangle - \langle \Phi^\infty_\varepsilon, 1_{[0,1]^2} \rangle \|_2 \leq \| \langle \Phi^\infty, 1_{[0,1]^2} \rangle - \langle \Phi^\eta, 1_{[0,1]^2} \rangle \|_2 + \| \langle \Phi^\eta, 1_{[0,1]^2} \rangle - \langle \Phi^\infty_\varepsilon, 1_{[0,1]^2} \rangle \|_2,
\]

where \( \Phi^\eta := \sum_j \sigma_j \mu^\eta_{C_j} \) denotes the lattice field and \( \Phi^\eta_\varepsilon := \sum_{j : \text{diam}(C_j) > \varepsilon} \sigma_j \mu^\eta_{C_j} \) is the lattice field with a cutoff on the diameter of clusters. Note that the normalizing factor used in [21] to define the normalized lattice field is the same as the normalizing factor used in this chapter to define the normalized counting measures for FK-Ising clusters.
As \( \eta \to 0 \), the first term in the right hand side of the last inequality tends to zero by Theorem 2.6 of [21]. For fixed \( \varepsilon > 0 \), the last term can be expressed as a finite sum, containing the normalized counting measures of clusters of diameter larger than \( \varepsilon \) that intersect the unit square. As \( \eta \to 0 \), this term tends to zero because of the convergence in probability of normalized counting measures proved in Theorem 5.7.2 under Assumption IV, and the \( L^3 \) bounds provided by Lemma 5.3.15.

The remaining term can be made arbitrarily small by letting \( \eta \to 0 \) and taking \( \varepsilon \) small. This follows from results and calculations in [24]. For a proof of this statement, see the proof of Proposition 6.2 of [19]. This concludes the proof of the theorem. \( \square \)

We remark that there has been recent progress [52, 29] on the full scaling limit of the critical Ising model in bounded domains with, say, plus boundary condition, corresponding to wired boundary condition for the FK-Ising model. Such a scaling limit is supposed to be unique and conformally invariant. Assuming that, the results and methods in this paper would be sufficient to prove conformal invariance/covariance away from the boundary. More precisely, assuming the uniqueness and conformal invariance of the full scaling limit in terms of loops for the critical FK-Ising percolation in a bounded domain \( D \) with wired boundary condition, our results and methods would imply that the collection of FK-Ising clusters completely contained in some smaller domain \( D' \subset D \), with \( \partial D' \) at positive distance from \( \partial D \), has a conformally invariant scaling limit. Analogously, the corresponding collection of counting measures would be conformally covariant. In order to get a full analogue of Theorem 5.2.2, one would need additional arguments to deal with the wired boundary condition on \( \partial D \).

### 5.3 Further notation and preliminaries

In the above we interpreted the union of red hexagons in a percolation configuration \( \sigma_\eta \), as a (random) subset of \( \mathbb{C} \). In the following, as an intermediate step, we will consider a percolation configuration as a (random) collection of loops. These loops form the boundaries of the clusters. We will describe this space first. In order to define the clusters as subsets of the plane, we will also consider the (random) collection of quads (‘topological squares’ with two marked opposing sides) which are crossed horizontally. This leads us to the Schramm–Smirnov [79] topological space, which we briefly recall in the second subsection.

#### 5.3.1 Space of nonsimple Loops

The random collection of loops will be denoted by \( L_\eta \) for \( \eta \geq 0 \). The distance between two curves \( l,l' \) is defined as

\[
\begin{align*}
    d_c(l,l') := \inf_{t \in [0,1]} \sup_{t \in [0,1]} \Delta(l(t),l'(t)),
\end{align*}
\]

where the infimum is over all parametrizations of the curves. The distance between closed sets of curves is defined similarly to the distance between collections of subsets of the Riemann sphere \( \hat{\mathbb{C}} \). The space of closed sets of loops is a complete separable metric space.
For $\eta > 0$ the boundaries of the red clusters in $\sigma_\eta$ is the closed set of loops, denoted by $L_\eta$. This set converges in distribution to $L_0$, called the \textit{continuum nonsimple loop process}.

### 5.3.2 Space of quad-crossings

We borrow the notation and definitions from [38]. Let $D \subset \hat{\mathbb{C}}$ be open. A quad $Q$ in $D$ is a homeomorphism $Q : [0, 1]^2 \to Q([0, 1]^2) \subseteq D$. Let $Q_D$ be the set of all quads, which we equip with the supremum metric

$$d(Q_1, Q_2) = \sup_{z \in [0, 1]^2} |Q_1(z) - Q_2(z)|$$

for $Q_1, Q_2 \in Q_D$.

A crossing of a quad $Q$ is a closed connected subset of $Q\left([0, 1]^2\right)$ which intersects $Q\left(\{0\} \times [0, 1]\right)$ as well as $Q\left(\{1\} \times [0, 1]\right)$. The crossings induce a natural partial order denoted by $\leq$ on $Q_D$. We write $Q_1 \leq Q_2$ if all the crossings of $Q_2$ contain a crossing of $Q_1$. For technical reasons, we also introduce a slightly less natural partial order on $Q_D$: we write $Q_1 < Q_2$ if there are open neighbourhoods $N_i$ of $Q_i$ such that for all $N_i \in \mathcal{N}_i$, $i \in \{1, 2\}$, $N_1 \subseteq N_2$. We consider the collection of all closed hereditary subsets of $Q_D$ with respect to $\leq$ and denote it by $\mathcal{H}_D$. It is the collection of the closed sets $S \subset Q_D$ such that if $Q \in S$ and $Q' \in Q_D$ with $Q' \leq Q$ then $Q' \in S$.

For a quad $Q \in Q_D$ let $\square_Q$ denote the set

$$\square_Q := \{S \in \mathcal{H}_D \mid Q \in S\},$$

which corresponds with the configurations where $Q$ is crossed. For an open subset $U \subset Q_D$ let $\square_U$ denote the set

$$\square_U := \{S \in \mathcal{H}_D \mid U \cap S = \emptyset\},$$

which corresponds with the configurations where none of the quads of $U$ is crossed.

We endow $\mathcal{H}_D$ with the topology $\mathcal{T}_D$ which is the minimal topology containing the sets $\square_Q$ and $\square_U$ as open sets for all $Q \in Q_D$ and $U \subset Q_D$ open. We have:

\textbf{Theorem 5.3.1} (Theorem 1.13 of [79]). \textit{Let $D$ be an open subset of $\hat{\mathbb{C}}$. Then the topological space $(\mathcal{H}_D, \mathcal{T}_D)$ is a compact metrizable Hausdorff space.}

Using this topological structure, we construct the Borel $\sigma$-algebra on $\mathcal{H}_D$. We get:

\textbf{Corollary 5.3.2} (Corollary 1.15 of [79]). \textit{Prob$(\mathcal{H}_D)$, the space of Borel probability measures of $(\mathcal{H}_D, \mathcal{T}_D)$, equipped with the weak* topology is a compact metrizable Hausdorff space.}

\textit{Notational remarks 5.3.3.} i) In the following we abuse the notation of a quad $Q$.

When we refer to $Q$ as a subset of $\hat{\mathbb{C}}$, we consider its range $Q([0, 1]^2) \subset \hat{\mathbb{C}}$. 


ii) Note that a percolation configuration $\sigma_\eta$, as defined in the introduction, naturally induces a quad-crossing configuration $\omega_\eta \in \mathcal{H}_\hat{C}$, namely

$$\omega_\eta := \{Q \in Q_\hat{C} \mid \sigma_\eta \text{ contains a crossing of } Q\}.$$  \hfill (5.11)

Furthermore, $\mathbb{P}_\eta$ will denote the law governing $(\omega_\eta \times L_\eta)$.

Further we will need the following definitions for restrictions of the configuration to a subset of the Riemann Sphere.

**Definition 5.3.4.** Let $D \subseteq \hat{C}$ be an open set and $\omega \in \mathcal{H}_\hat{C}$. Then $\omega|_D$, the restriction of $\omega$ to $D$, is defined as

$$\omega|_D := \{Q \in \omega : Q \subset D\}.$$  

The image of $\omega|_D$ under a conformal map $f : D \to \hat{C}$ is defined as

$$f(\omega|_D) := \{f(Q) : Q \in \omega|_D\} \in \mathcal{H}_{f(D)}.$$ 

The restriction of the Loop process to $D$ is defined as

$$L|_D := \{l : \exists \tilde{l} \in L \text{ s.t. } l \text{ is an excursion of } \tilde{l} \text{ in } D\}.$$ 

The image of $L|_D$ under a conformal map $f : D \to \hat{C}$ is defined as

$$f(L|_D) := \{f(l) : l \in L|_D\}.$$ 

Furthermore, $\mathbb{P}_{\eta,D}$ denotes the law of $(\omega_{\eta,D}, L_{\eta,D}) := (\omega_{\eta}|_D, L_{\eta}|_D)$ for $\eta \geq 0$.

### 5.3.3 Assumptions

In the following we list the assumptions which are used throughout the article.

The edge set in the sublattice on $D \subset \mathbb{C}$ of $\eta \mathbb{L}$ is $(\eta E(\mathbb{L}))|_D := \{(u, v) \in \eta E(\mathbb{L}) : u, v \in \eta V(\mathbb{L}) \cap D\}$. The discrete boundary of $D \subset \mathbb{C}$ of the lattice $\eta \mathbb{L}$ is defined by:

$$\partial_\eta D := \{u \in \eta V(\mathbb{L}) \cap D : \exists v \in \eta \mathbb{L} : u \sim v \text{ and } v \in \eta \mathbb{L} \cap (\mathbb{C} \setminus D)\}.$$ 

A boundary condition $\xi$ is a partition of the discrete boundary of $D$. A set in this partition denotes the vertices which are connected via red hexagons or edges (depending on the model) in $\mathbb{C} \setminus D$. When $\xi$ is omitted, it means we are considering the full plane model and are not specifying any boundary conditions on the discrete boundary of $D$.

**Assumption I** (Domain Markov Property). Let $D \subset E \subset \mathbb{C}$ be open sets. Further let $S \subset \overline{E \setminus D}$ and $T \subset \overline{D}$ closed sets. Then

$$\mathbb{P}_\eta(\sigma_D = T \cap D \mid \sigma_{E \setminus \overline{D}} = S) = \mathbb{P}_\eta(\sigma_D = T \mid \xi) =: \mathbb{P}_\eta^{\xi}(\sigma_D = T)$$

where $\sigma_D = \sigma_\eta \cap D$ and $\xi$ is the discrete boundary condition on $D$ induced by $\sigma_{E \setminus \overline{D}} = S$. 
For some models the randomness is put on the vertices (e.g. Bernoulli site percolation) and for others on the edges (e.g. FK-Ising percolation). For the models of the first form we define $\Omega_{\eta,D} := \eta V(L) \cap D$ and for models of the second form $\Omega_{\eta,D} := (\eta E(L))|_D$.

**Assumption II** (Strong positive-association / FKG). The finite measures are strongly positively-associated. More precisely, let $D \subset \mathbb{C}$ be a bounded closed set. For every boundary condition $\xi$ on $\partial_{\eta} D$ and increasing functions $f, g : \{\text{red, blue}\}^{\Omega_{\eta,D}} \rightarrow \mathbb{R}$, we have

$$E_{\eta}^{\xi}[f \cdot g] \geq E_{\eta}^{\xi}[f] \cdot E_{\eta}^{\xi}[g].$$

Hence for increasing events $A, B$ and boundary condition $\xi$ on $\partial_{\eta} D$:

$$\mathbb{P}_{\eta}^{\xi}(A \cap B) \geq \mathbb{P}_{\eta}^{\xi}(A) \mathbb{P}_{\eta}^{\xi}(B).$$

It is well known that monotonicity in the boundary condition is equivalent to strongly positively-association, if the measure is strictly positive (has the finite energy property), i.e. every configuration has strictly positive probability. (See e.g. [42, Theorem 2.24].) Furthermore it is well known that positive association survives the limit as the lattice grows towards infinity. See for example [42, Proposition 4.10].

In the following assumption $l(Q)$ denotes the extremal length of $Q$, that is, let $\phi : Q \rightarrow [0,a] \times [0,1]$ conformal such that $\phi(Q(\{0\} \times [0,1])) = \{0\} \times [0,1]$ and $\phi(Q(\{1\} \times [0,1])) = \{a\} \times [0,1]$, then $l(Q) = a$.

**Assumption III** (RSW). Let $M > 0$. There exist $\delta > 0$ such that, for every quad $Q$ with $l(Q) \leq M$ and every boundary condition $\xi$ on the discrete boundary of $Q([0,1]^2)$:

$$\mathbb{P}_{\eta}^{\xi}(\omega_{\eta} \in \Box Q) \geq \delta$$

and for every quad $Q$ with $l(Q) \geq M$ and every boundary condition $\xi$ on the discrete boundary of $Q([0,1]^2)$:

$$\mathbb{P}_{\eta}^{\xi}(\omega_{\eta} \not\in \Box Q) \leq 1 - \delta.$$

**Assumption IV** (Full Scaling Limit). As $\eta \rightarrow 0$, the law of $L_{\eta}$ converges weakly to a random infinite collection of loops $L_0$ in the induced Hausdorff metric on collections of loops induced by the distance (5.10). Moreover, the limiting law is conformally invariant.

### 5.3.4 Arm events

For $S \subset \hat{\mathbb{C}}$, let $\partial S, \text{int}(S), \bar{S}$ denote the boundary, interior and the closure of $S$, respectively. We call the elements of $\{0,1\}^k$, $k \geq 0$ as colour-sequences. For ease of notation, we omit the commas in the notation of the colour sequences, e.g. we write (101) for $(1,0,1)$.

**Definition 5.3.5.** Let $l \in \mathbb{N}$, $\kappa \in \{0,1\}^l$, $S \subseteq \hat{\mathbb{C}}$ and $D, E$ be two disjoint open, simply connected subsets of $\hat{\mathbb{C}}$ with piecewise smooth boundary. Let $D \leftrightarrow_{\kappa,S} E$ denote the event that there are $\delta > 0$ and quads $Q_i \in Q_S$, $i = 1, 2, \ldots, l$ which satisfy the following conditions.
Chapter 5. Conformal measure ensembles for percolation and FK-Ising

1. \( \omega \in \square Q_i \) for \( i \in \{1, 2, \ldots, l\} \) with \( \kappa_i = 1 \) and \( \omega \in \square Q_i \) for \( i \in \{1, 2, \ldots, l\} \) with \( \kappa_i = 0 \).

2. For all \( i \neq j \in \{1, 2, \ldots, l\} \) with \( \kappa_i = \kappa_j \), the quads \( Q_i \) and \( Q_j \), viewed as subsets of \( \hat{C} \), are disjoint, and are at distance at least \( \delta \) from each other and from the boundary of \( S \).

3. \( \Lambda_\delta + Q_i (\{0\} \times [0, 1]) \subset D \) and \( \Lambda_\delta + Q_i (\{1\} \times [0, 1]) \subset E \) for \( i \in \{1, 2, \ldots, l\} \) with \( \kappa_i = 1 \).

4. \( \Lambda_\delta + Q_i ([0, 1] \times \{0\}) \subset D \) and \( \Lambda_\delta + Q_i ([0, 1] \times \{1\}) \subset E \) for \( i \in \{1, 2, \ldots, l\} \) with \( \kappa_i = 0 \).

5. The intersections \( Q_i \cap D \), for \( i = 1, 2, \ldots, l \), are at distance at least \( \delta \) from each other, the same holds for \( Q_i \cap E \).

6. A counterclockwise order of the quads \( Q_i \), \( i = 1, 2, \ldots, l \), is given by ordering counterclockwise the connected components of \( Q_i \cap D \) containing \( Q_i(0, 0) \).

When the subscript \( S \) is omitted, it is assumed to be \( \hat{C} \).

Remark 5.3.6. It is a simple exercise to show that the events \( D \xleftarrow{\kappa, S} E \) are Borel(\( \mathcal{F}_\hat{C} \))-measurable. See [38, Lemma 2.9] for more details.

In the following we consider some special arm events. For \( z \in \mathbb{C}, a > 0 \) let \( H_1(z, a), H_2(z, a), H_3(z, a), H_4(z, a) \) denote the left, lower, right, and upper half planes which have the right, top, left and bottom sides of \( \Lambda_a(z) \) on its boundary, respectively. For \( z \in \mathbb{C}, 0 < a < b \) we set

\[
A(z; a, b) := \Lambda_b(z) \setminus \Lambda_a(z).
\]

Furthermore, for \( i = 1, 2, 3, 4, \kappa \in \{0, 1\}^l \) and \( \kappa' \in \{0, 1\}^{l'} \) with \( l, l' \geq 0 \) we define the event where there are \( l + l' \) disjoint arms with colour-sequence \( \kappa \vee \kappa' := (\kappa_1, \ldots, \kappa_l, \kappa'_1, \ldots, \kappa'_{l'}) \) in \( A(z; a, b) \) so that the \( l' \) arms, with colour-sequence \( \kappa' \), are in the half-plane \( H_i(z, a) \). That is,

\[
A^{i, \kappa, \kappa'}_{\kappa', \kappa'}(z; a, b) := \left\{ \Lambda_a(z) \xleftarrow{\kappa, \kappa'} \left( \hat{C} \setminus \Lambda_b(z) \right) \right\} \cap \left\{ \Lambda_a(z) \xleftarrow{\kappa', H_i(z, a)} \left( \hat{C} \setminus \Lambda_b(z) \right) \right\}
\]

(5.12)

In the notation above, when \( z \) is omitted, it is assumed to be 0.

Finally, for \( 0 < a < b \) and boundary condition \( \xi \) on \( \partial_{\eta} \Lambda_b \) we set

\[
\pi_{1, \xi}^{\eta}(a, b) := \mathbb{P}_{\eta}^{\xi}(A^1_{(1), \emptyset}(a, b)), \quad \pi_{4, \xi}^{\eta}(a, b) := \mathbb{P}_{\eta}^{\xi}(A^1_{(1010), \emptyset}(a, b)),
\]

\[
\pi_{0, \xi}^{\eta}(a, b) := \mathbb{P}_{\eta}^{\xi}(A^1_{(0101), \emptyset}(a, b)), \quad \pi_{0, 3}^{\eta}(a, b) := \mathbb{P}_{\eta}^{\xi}(A^1_{\emptyset, (010)}(a, b)),
\]

\[
\pi_{1, 3}^{\eta}(a, b) := \mathbb{P}_{\eta}^{\xi}(A^1_{(1), (010)}(a, b)).
\]

Remark 5.3.7. The (technical) reason to define \( H_i(z, a) \) in this slightly unnatural way, will become clear in the proof of Lemma 5.4.7.
5.3.5 Consequences of RSW

Lemma 5.3.8 (Quasi multiplicativity). Suppose that Assumptions I-III hold. There is a constant $C > 0$ so that

$$\mathbb{P}^{\xi}_{\eta}(A^{1}(a,b)) \leq C \frac{\pi^{\eta,\xi}_{1}(a,c)}{\pi^{\eta,\xi}_{1}(b,c)}$$

for all $a, b, c, \eta > 0$ with $\eta < a < b < c$ and boundary condition $\xi$ on $\partial_{\eta} \Lambda_{c}$.

Lemma 5.3.9. Suppose that Assumptions I-III hold. There are constants $\lambda_{1} \in (0,1)$ and $C > 0$ so that

$$\pi^{\eta,\xi}_{1}(\eta, b) \geq C \left( \frac{a}{b} \right)^{\lambda_{1}} \mathbb{P}^{\xi}_{\eta}(A^{1}(\eta, a))$$

for all $b > a > \eta$ and boundary condition $\xi$ on $\partial_{\eta} \Lambda_{b}$.

Lemma 5.3.10. Suppose that Assumptions I-III hold. There are positive constants $C, \lambda_{6}$ such that

$$\pi^{\eta,\xi}_{6}(a, b) \leq C \left( \frac{a}{b} \right)^{2 + \lambda_{6}}, \quad \pi^{\eta,\xi}_{0,3}(a, b) \leq C \left( \frac{a}{b} \right)^{2} \quad (5.13)$$

for all $0 < \eta < a < b$ and boundary condition $\xi$ on $\partial_{\eta} \Lambda_{b}$.

Lemma 5.3.11. Suppose that Assumptions I-III hold. There are positive constants $C, \lambda_{1,3}$ such that

$$\pi^{\eta,\xi}_{1,3}(a, b) \leq C \left( \frac{a}{b} \right)^{2 + \lambda_{1,3}} \quad (5.14)$$

for all $0 < \eta < a < b$ and boundary condition $\xi$ on $\partial_{\eta} \Lambda_{b}$.

Lemma 5.3.12. Suppose that Assumptions I-III hold. There are constants $C, \lambda > 0$ so that

$$\frac{\pi^{\eta,\xi}_{1}(a, b)}{\pi^{\eta,\xi}_{4}(a, b)} \geq C \left( \frac{b}{a} \right)^{\lambda}$$

for all $b > a > \eta$ and boundary condition $\xi$ on $\partial_{\eta} \Lambda_{b}$.

For the sake of generality, we have stated the bounds in the previous lemmas in the presence of boundary conditions. However, in the rest of this chapter only the full-plane versions of the bounds will appear, so the superscript $\xi$ will be dropped. (The versions with boundary conditions are necessary to obtain results that we use in this paper, but whose proofs we do not reproduce.) For the next lemma we need some additional notation.

Definition 5.3.13. For $\eta, a > 0$ let

$$V^{\eta}_{a} := \{ v \in \Lambda_{a/2} \cap \eta V \mid v \xrightarrow{\eta} \partial \Lambda_{a} \text{ in } \omega_{\eta} \}$$

denote the number of vertices in $\Lambda_{a/2}$ connected to $\partial \Lambda_{a}$ in $\sigma_{\eta}$. 

Lemma 5.3.14. Suppose that Assumptions I-III hold. Then there are positive constants $c,C$ such that
\[ \mathbb{P}_\eta(|V_a^\eta| \geq x(a/\eta)^2 \pi_1^\eta(\eta,a)) \leq Ce^{-cx} \]
for all $a > \eta$ and $x \geq 0$.

Lemma 5.3.15. Suppose that Assumptions I-III hold. Then there is a constant $C > 0$ such that
\[ \mathbb{E}_\eta[|W_a^\eta|^3] \leq C\eta^{-6}\pi_1^\eta(\eta,a)^{-3} \]
for all $0 < \eta < a < 1/2$, where
\[ W_a^\eta := \{ v \in \Lambda_1 \cap \eta V \mid v \leftrightarrow \partial \Lambda_a(v) \text{ in } \omega_\eta \}. \]

Proof of Lemmas 5.3.8 - 5.3.15. Lemmas 5.3.10 and 5.3.11 follow from Assumptions I - III, as explained in e.g. [65, 41] for the case of Bernoulli percolation and in [28, Corollary 1.5 and Remark 1.6] for the case of FK-Ising percolation. (The additional boundary conditions, which are not present in the above mentioned corollary and remark in [28], do not affect the results. This can easily be deduced from equation (5.1) in [28].)

Also Lemmas 5.3.9 and 5.3.12 follow from standard RSW, FKG arguments.

Lemma 5.3.8 is similar to [28, Theorem 1.3], which is shown to follow from our assumptions I-III. The boundary condition on $\partial_\eta \Lambda_e$ has no effect on the proof, because the RSW result is uniform in the boundary conditions. (Furthermore there is no need to "make" the arms well separated on $\partial_\eta \Lambda_e$.)

An easy proof of Lemma 5.3.14 for critical percolation can be found in [64]. It is easy to see that the same proof can be modified in such a way that the result follows from Lemmas 5.3.8 - 5.3.12, and hence from Assumptions I-III. For percolation, Lemma 5.3.14 can also be found in [16, Lemma 6.1], and for FK-Ising percolation in [21, Lemma 3.10].

Finally Lemma 5.3.15 can be proved easily using Lemma 5.3.8. See for example [38, Lemma 4.5] or the proof of Lemma 5.3.14. \qed

5.3.6 Additional preliminaries

Lemma 5.3.16. Suppose that Assumptions I-IV hold. The set of crossed quads is, almost surely, measurable with respect to the collection of loops.

Proof of Lemma 5.3.16. A proof of this can be found in [38, Section 2.3] and follows almost immediately from arguments given in [22, Section 5.2]. The proof of the measurability of quad crossings with respect to the collection of loops makes use of three properties of the loop process, which all follow from RSW techniques (see the first three items of Theorem 3 in [22, Section 5.2]). Because of this, the measurability is a simple consequence of our Assumptions I-IV. \qed

Remark 5.3.17. Assumption IV together with the separability of $\mathcal{H}_C$ shows that there is a coupling $\mathbb{P}$ so that $\omega_\eta \to \omega_0$ a.s. as $\eta \to 0$. 
Before we proceed to the next lemma, we recall the following result on the scaling limits of arm events. A slightly weaker version of the following lemma appeared as [38, Lemma 2.9]. Its proof extends immediately to the more general case.

**Lemma 5.3.18 (Lemma 2.9 of [38]).** Suppose that Assumptions I-IV hold. Then, under a coupling $\mathbb{P}$ of $(\mathbb{P}_\eta)_{\eta \geq 0}$ such that $\omega_\eta \to \omega_0$ almost surely, we have for events $\mathcal{D} \in \{ \{ A \leftarrow (1), S \right\} \to B \}, \{ A \leftarrow (01), S \} \to B \}$, $A_{\kappa,\kappa'}^i (z; a, b)$,

$$1_{\mathcal{D}}(\omega_\eta) \to 1_{\mathcal{D}}(\omega_0) \quad \text{in } \mathbb{P}\text{-probability},$$

for $(\kappa, \kappa') \in \{ ((1), 0), ((1010), 0), ((010101), 0), (0, (010)), ((1), (010)) \}$, rectangle $S \subseteq \mathbb{C}$, $i \in \{ 1, 2, 3, 4 \}$, $0 < a < b$ and $A, B$ disjoint open subsets of $\mathbb{C}$ with piece-wise smooth boundary.

The lemma above implies that for all $a, b > 0$ with $a < b$ the probability $\pi_1^\eta(a, b)$ converges as $\eta \to 0$. We write $\pi_1^\eta(a, b)$ for the limit. General arguments [8, Section 4] using Lemma 5.3.8 above show that

$$\pi_1^0(a, b) = \left( \frac{a}{b} \right)^{\alpha_1 + o(1)}$$

for some $\alpha_1 \geq 0$ where $o(1)$ is understood as $b/a \to \infty$. Lemma 5.3.9 shows that $\alpha_1 < 1$.

We need some additional notation for the next theorems. For $z \in \mathbb{C}$ and $a > 0$ let $\Lambda_a'(z) := \{ u \in \mathbb{C} \mid \Re(u - z), \Im(u - z) \in [-a, a] \}$. Note that $\Lambda_a(z)$ and $\Lambda_a'(z)$ differ only on their boundary. For an annulus $A = A(z; a, b)$ let

$$\mu_{1,A}^\eta := \frac{\eta}{\pi_1^\eta(\eta, 1)} \sum_{v \in \Lambda_a'(z) \cap \eta V} \delta_v \mathbf{1}\{ v \leftarrow \partial \Lambda_b(z) \in \omega_\eta \}$$

(5.16)

denote the counting measure of the vertices in $\Lambda_a'(z)$ with an arm to $\partial \Lambda_b(z)$ at scale $\eta$.

**Theorem 5.3.19.** Suppose that Assumptions I-IV hold. Let $A = A(z; a, b)$ be an annulus, and $\mathbb{P}$ be a coupling such that $\omega_\eta \to \omega_0$ a.s. as $\eta \to 0$. Then the measures $\mu_{1,A}^\eta$ converge weakly to $\mu_{1,A}^0$ in probability under the coupling $\mathbb{P}$ as $\eta$ tends to 0. Furthermore, $\mu_{1,A}^0$ is a measurable function of $\omega_0$. In particular, the pair $(\omega_\eta, \mu_{1,A}^\eta)$ converges to $(\omega_0, \mu_{1,A}^0)$ in distribution as $\eta \to 0$.

Theorem 5.3.19 is proved for site percolation on the triangular lattice in [38] where it is Theorem 5.1. Namely, it is easy to check that the proof of [38, Theorem 5.1] shows that the measures $\mu_{1,A}^\eta \overset{\mathbb{P}}{\to} \mu_{1,A}^0$ under the coupling $\mathbb{P}$ converge weakly in probability as $\eta \to 0$. For FK-Ising, a sketch proof for a theorem similar to this was given in [21]. Unfortunately the proof contains a mistake, but luckily the mistake can be easily fixed. Below we give an informal sketch of the proof of Theorem 5.3.19, following the proof in [21] and briefly explaining how to fix it.

The strategy is to approximate, in the $L^2$-sense, the one-arm measure by the number of mesoscopic boxes connected to $\partial \Lambda_b(z)$, multiplied by a constant depending
on the size of the boxes. Here mesoscopic means much larger than the mesh size $\eta$ but much smaller than $a$.

In order to get $L^2$-bounds on the error terms, first we use a coupling argument to argue that the boxes which are far away from each other are almost independent. Namely, with high probability one can draw a red circuit around one of the boxes, which is also conditioned on having a long red arm (because of positive association, that event can only increase the probability of a red circuit). This red circuit makes, via the Domain Markov Property, the contribution of the surrounded box independent of that of the other boxes. The total contribution of the boxes which are close to each other is negligible. Secondly we use a ratio limit argument, based on the existence of the one-arm exponent $\alpha_1$ from (5.15), to show that the contribution of a single box is approximately a constant, which only depends on the size of the mesoscopic box.

The small mistake in [21] mentioned above is in the assumption that the convergence in Lemma 5.3.18 is almost sure, as claimed in an earlier version of [38]. However, as noted in the final version of [38], one can only prove convergence in probability. Luckily, arguments in [38] show that convergence in probability, together with $L^2$ bounds from Lemma 5.3.15, is sufficient to prove convergence in $L^2$ of the number of mesoscopic boxes connected to $\partial\Lambda_b(z)$ times a constant depending on the size of these boxes.

5.3.7 Validity of the assumptions

The case of critical percolation

Now we check that the Assumptions above hold for critical site percolation on the triangular lattice.

**Theorem 5.3.20.** For critical site percolation on the triangular lattice, the Assumptions I-IV hold.

**Proof of Theorem 5.3.20.** The Domain Markov Property, Assumption I, is trivial, one even has independence. Assumption II is well known, see e.g. [42, Theorem 3.8]. RSW, Assumption III, is also well known, see for example [41, 63].

The existence of the full scaling limit in Assumption IV is proved by Camia and Newman in [22]. The value of $\alpha_1$ is $5/48$ as proved in [61].

The case of FK-Ising model

The Domain Markov Property and strongly positive association are standard and well known see e.g. [42]. The recent development of the RSW theory for the FK-Ising model proves Assumption III. Namely it follows from Theorem 1.1 in [28] combined with the fact that the discrete extremal length, used in [28] is comparable to its continuous counterpart, used here, see [27, Proposition 6.2].

Unfortunately, to our knowledge, Assumption IV has not yet been proved for the FK-Ising model. The fundamental reason is that the analogue of the results in [22] is missing, in particular, the uniqueness of the full scaling limit has not yet been proved for the FK-Ising model. The value of $\alpha_1$ for the Ising model is $1/8$. As shown in [24], this can be seen from the behaviour of the Ising two-point function at criticality [83].
5.4 Approximations of large clusters

In the following we give two approximations of open clusters with diameter at least \( \delta > 0 \), which are completely contained in \( \Lambda_k \). The first one relies solely on the arm events described in the previous section, while the other is ‘the natural’ one, namely it is simply the union of \( \varepsilon \)-boxes which intersect the cluster. The advantage of the first approximation is that it can also be defined in the limit as the mesh size goes to 0. First we prove Proposition 5.4.3, which shows that on a certain event these two approximations coincide. Then in Section 5.4.1 we give a lower bound for the probability of the event above.

For simplicity, we set \( k = 1 \) from now on. The constructions and proofs for different values of \( k \) are analogous. Let \( \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \). For \( \varepsilon > 0 \), let \( B_\varepsilon \) be the following collection of squares of side length \( \varepsilon \):

\[
B_\varepsilon := \{ \Lambda_{\varepsilon/2}(\varepsilon z) \mid z \in \Lambda_{[1/\varepsilon]} \cap \mathbb{Z}[i] \}.
\]

Fix \( \omega \in \mathcal{H}_\varepsilon \). We define the graph \( G_\varepsilon = G_\varepsilon(\omega) \) as follows. Its vertex set is \( B_\varepsilon \). The boxes \( \Lambda_{\varepsilon/2}(\varepsilon z), \Lambda_{\varepsilon/2}(\varepsilon z') \in B_\varepsilon \) are connected by an edge if \( ||z - z'||_\infty = 1 \) or if \( \omega \in \{ \Lambda_{\varepsilon/2}(\varepsilon z) \leftrightarrow \Lambda_{\varepsilon/2}(\varepsilon z') \} \). For a graph \( H \) with \( V(H) \subseteq B_\varepsilon \) we set

\[
U(H) := \bigcup_{\Lambda \in V(H)} \Lambda \subseteq \Lambda_{1+2\varepsilon}.
\]  (5.17)

Let \( L(H) \) denote the set of leftmost vertices of \( H \). That is,

\[
L(H) := \{ \Lambda_{\varepsilon/2}(\varepsilon z) \in V(H) \mid \forall z' \in \mathbb{Z}[i] \text{ with } \Lambda_{\varepsilon/2}(\varepsilon z') \in V(H) \text{ we have } \Re z \leq \Re z' \}.
\]

Similarly, we define \( R(H), T(H), B(H) \) as the right-, top- and bottommost vertices of \( H \), respectively. Let \( SH(H) \) (resp. \( SV(H) \)) denote the most narrow double infinite horizontal (resp. vertical) strip containing \( U(H) \). Finally, let \( SR(H) \) denote the smallest rectangle containing \( U(H) \) with sides parallel to one of the axes. Thus \( SR(H) = SH(H) \cap SV(H) \).

**Definition 5.4.1.** For \( z, z' \in \mathbb{C} \), we set \( \text{dist}_1(z, z') = |\Re(z - z')| \) and \( \text{dist}_2(z, z') = |\Im(z - z')| \). We call \( \text{dist}_1, \text{dist}_2 \) as the distance in the horizontal and vertical directions, respectively. We also use the notation \( d_\infty(z, z') := ||z - z'||_\infty = \text{dist}_1(z, z') + \text{dist}_2(z, z') \) for the \( L^\infty \) distance.

For disjoint sets \( A, B \subset \hat{\mathbb{C}} \) we set \( \text{dist}_i(A, B) := \inf \{ \text{dist}_i(z, z') : z \in A, z' \in B \} \) for \( i = 1, 2 \).

Let \( \eta > 0 \), \( \Lambda = \Lambda_{\varepsilon/2}(z) \in B_\varepsilon \) and \( \Lambda' = \Lambda_{\varepsilon/2}(z') \in B_\varepsilon \). Suppose there is a cluster which is completely contained in \( \Lambda_1 \), such that \( \Lambda \) contains a leftmost vertex of this cluster and \( \Lambda' \) a rightmost vertex. Then \( \Lambda \) and \( \Lambda' \) are connected by 2 blue arms and one red arm in between them.

This leads us to the following definition, which gives us a way to characterize the clusters using only arm events.

**Definition 5.4.2.** Let \( \omega \in \mathcal{H}_\varepsilon \) and \( G_\varepsilon = G_\varepsilon(\omega) \) the graph defined above. Let \( H \) be a subgraph of \( G_\varepsilon(\omega) \). We say that \( H \) is good, if it satisfies the following conditions.
1. $H$ is complete,

2. $U(H) \subseteq \Lambda_1$,

3. $H$ is maximal, that is, if $\Lambda \in V(G_\varepsilon)$ and $(\Lambda, \Lambda') \in E(G_\varepsilon)$ for all $\Lambda' \in V(H)$, then $\Lambda \in V(H)$,

4. $\text{diam}(U(H)) \geq \delta$,

5. for all $\Lambda \in L(H)$ and $\Lambda' \in R(H)$ we have $\omega \in \{\Lambda \xrightarrow{(010)\cdot SV(H)} \Lambda'\}$, a similar condition holds for $\Lambda \in T(H)$ and $\Lambda' \in B(H)$, with $SV(H)$ replaced by $SH(H)$.

For a set $S \subseteq \mathbb{C}$ and $\varepsilon > 0$ let $K_\varepsilon(S)$ denote the complete graph on the vertex set

$$\{\Lambda_{\varepsilon/2}(\varepsilon z) \mid z \in \mathbb{Z}[i] \text{ and } \Lambda_{\varepsilon/2}(\varepsilon z) \cap S \neq \emptyset\}.$$ 

Further, we use the shorthand

$$U_\varepsilon(S) := U(K_\varepsilon(S)) = \bigcup_{z \in \mathbb{Z}[i]: \Lambda_{\varepsilon/2}(\varepsilon z) \cap S \neq \emptyset} \Lambda_{\varepsilon/2}(\varepsilon z).$$

For $C_\eta \in \mathcal{C}_1(\delta)$, the graph $K_\varepsilon(C_\eta)$ approximates $C_\eta$ in the sense that $d_H(C_\eta, U_\varepsilon(C_\eta)) < \varepsilon$. This is the second approximation of large clusters we referred to in the beginning of this section. Our next aim is to find an event where the two approximations coincide.

In the following we use the quantities defined above in the case where $\omega = \omega_\eta$ for some $\eta \geq 0$. We denote the particular choice of $\eta$ in the superscript, for example $G_\varepsilon^\eta := G_\varepsilon(\omega_\eta)$. We shall prove:

**Proposition 5.4.3.** Let $\eta, \varepsilon, \delta > 0$ with $1/10 > \delta > 10\varepsilon$. Suppose that $\omega_\eta \in \mathcal{E}(\varepsilon, \delta)$, where $\mathcal{E}(\varepsilon, \delta)$ as in (5.18) below.

i) Then for all good subgraphs $H \leq G_\varepsilon^\eta$ there is a unique cluster $C^\eta \in \mathcal{C}_1(\delta)$ such that $H = K_\varepsilon(C^\eta)$.

ii) Conversely, if $C^\eta \in \mathcal{C}_1(\delta)$, then $K_\varepsilon(C^\eta)$ is a good subgraph of $G_\varepsilon^\eta$.

**Proof of Proposition 5.4.3.** Proposition 5.4.3 follows from the combination of Lemma 5.4.5 and 5.4.7 with the definition (5.18) below. \qed

For $\varepsilon, \delta > 0$ we define the event as the intersection

$$\mathcal{E}(\varepsilon, \delta) := \mathcal{N}A(\varepsilon, \delta) \cap \mathcal{N}C(\varepsilon, \delta). \quad (5.18)$$

First we define the event $\mathcal{N}C(\varepsilon, \delta)$ below, then we introduce $\mathcal{N}A(\varepsilon, \delta)$ in Definition 5.4.6.

**Definition 5.4.4.** Let $0 < 10\varepsilon < \delta < 1$. We write $\mathcal{N}C(\varepsilon, \delta)^c$ for the union of events

$$A_{\theta,(010)}^j(z; \varepsilon/2, \delta/2 - 3\varepsilon) \cap A_{\theta,(010)}^{3j+2}(z'; \varepsilon/2, \delta/2 - 3\varepsilon) \quad (5.19)$$

for $j = 1, 2$, and squares $\Lambda_{\varepsilon/2}(z), \Lambda_{\varepsilon/2}(z') \in B_\varepsilon$ with $\text{dist}_j(z, z') \in (\delta - 3\varepsilon, \delta + 3\varepsilon)$. \hfill \square
Definition 5.4.4 implies the following lemma, which illuminates the choice of the event \( \mathcal{NC}(\varepsilon, \delta) \).

**Lemma 5.4.5.** Let \( 0 < 10\varepsilon < \delta < 1 \). On \( \omega \in \mathcal{NC}(\varepsilon, \delta) \) there is no cluster \( C^n \), which is completely contained in \( \Lambda_1 \) with diameter between \( \delta - 2\varepsilon \) and \( \delta \).

We define the event \( \mathcal{NA}(\varepsilon, \delta) \) which will be crucial in the following.

**Definition 5.4.6.** Let \( \varepsilon, \delta \) with \( 0 < 10\varepsilon < \delta < 1 \). We set \( \mathcal{NA}_1(\varepsilon, \delta) \) for the complement of the event

\[
\bigcup_{z \in \Lambda(\varepsilon, \delta) \cap \mathbb{Z}^2} \bigcup_{j=1}^4 A_{1,(010)}(\varepsilon z; \varepsilon/2, \delta/2 - 3\varepsilon).
\]

We write \( \mathcal{NA}_2(\varepsilon, \delta)^c \) for the union of events

\[
A_{0,(010)}(\varepsilon z; \varepsilon/2, \delta/2 - 3\varepsilon)
\]

for \( j = 1, 2, 3, 4 \), and squares \( \Lambda_{\varepsilon}/2(z) \in B \) with \( \min_{i \in \{1, 2\}} \text{dist}_i(\Lambda_{\varepsilon}/2(z), \partial \Lambda_k) \leq \varepsilon \). We define \( \mathcal{NA}(\varepsilon, \delta) := \mathcal{NA}_1(\varepsilon, \delta) \cap \mathcal{NA}_2(\varepsilon, \delta) \).

**Lemma 5.4.7.** Let \( \eta, \varepsilon, \delta > 0 \) with \( 0 < 10\varepsilon < \delta < 1 \). Suppose that \( \omega \in \mathcal{NA}(\varepsilon, \delta) \). We have:

i) If \( C^n \in \mathcal{C}_1^n(\delta) \), then \( K_\varepsilon(C^n) \) is a good subgraph of \( G_\varepsilon^n \).

ii) Conversely, for any good subgraph \( H \leq G_\varepsilon^n \), there is a unique cluster \( C^n \in \mathcal{C}_1^n(\delta - 2\varepsilon) \) such that \( H = K_\varepsilon(C^n) \).

**Proof of Lemma 5.4.7.** Let \( \varepsilon, \delta \) as in the lemma above, and \( \omega \in \mathcal{NA}(\varepsilon, \delta) \). First we prove part i) above. Apart from conditions (2) and (3), the conditions in Definition 5.4.2 are trivially satisfied. The fact that \( \omega \in \mathcal{NA}_2(\varepsilon, \delta) \) implies that condition (2) is satisfied. We prove condition (3) by contradiction.

Suppose that condition (3) is violated. Then there is \( \Lambda \in V(G_\varepsilon^n) \setminus V(K_\varepsilon(C^n)) \) such that \( (\Lambda, \Lambda') \in E(G_\varepsilon^n) \) for all \( \Lambda' \in V(K_\varepsilon(C^n)) \).

We can assume that the diameter of \( C^n \) is realized in the horizontal direction. Take \( L \in L(K_\varepsilon(C^n)) \) and \( R \in R(K_\varepsilon(C^n)) \). Let \( \gamma \) denote a path in \( C^n \) connecting \( L \) and \( R \). We can further assume that \( \text{dist}_1(\Lambda, L) > \delta/2 - \varepsilon \). Note that \( \gamma \) is not connected to \( \Lambda \). However, \( \Lambda \) is connected to \( L \). Hence the blue boundary of \( C^n \) separates \( \gamma \) from the connection between \( \Lambda \) and \( L \). We get, from \( L \) to distance \( \delta/2 - \varepsilon \), three half plane arms with colour sequence (010), and a fourth red arm from the connection between \( \Lambda \) and \( L \). In particular, \( \omega \in \mathcal{NA}_1(\varepsilon, \delta)^c \), we deduce part i) of Lemma 5.4.7.

Now we proceed to the proof of part ii). We may assume that the diameter of \( U(H) \) is realized between a leftmost and a rightmost point of it. Let \( L \in L(H) \), \( R \in R(H) \) and \( \gamma \) be a path in \( SR(H) \) connecting \( L \) and \( R \). Furthermore, let \( \Lambda' \in V(G_\varepsilon^n) \) be such that \( \gamma \) is connected to \( \Lambda' \) by a path in \( \sigma \cap \Lambda_1 \).

We show that \( (\Lambda, \Lambda') \in E(G_\varepsilon^n) \) for all \( \Lambda \in V(H) \). Suppose the contrary, i.e., there is \( \Lambda \in V(H) \) such that \( (\Lambda, \Lambda') \notin E(G_\varepsilon^n) \). Then \( \Lambda \) is not connected to \( \gamma \). Furthermore, we may assume that \( \text{dist}_1(\Lambda, L) > \delta/2 - \varepsilon \). Then as above, we find three half plane
arms with colour sequence \((010)\) and a fourth red arm starting at \(L\) to distance \(\delta/2-\varepsilon\). In particular, \(\omega_\eta \in \mathcal{N}A_1(\varepsilon, \delta)^c\), which contradicts the assumption on \(\omega_\eta\) above.

Hence \(\mathcal{A}' \in V(H)\) since \(H\) is maximal. Thus \(K_\varepsilon(C_\eta^\delta(\gamma)) \leq H\), where \(C_\eta^\delta(\gamma)\) denotes the connected component of \(\gamma\) in \(\sigma_\eta\). Note that \(K_\varepsilon(C_\eta^\delta(\gamma))\) is a good subgraph, since it satisfies condition 4 since \(\text{dist}_1(L, R) > \delta\), condition 3 by part i) of Lemma 5.4.7. This completes the proof of part ii) and that of Lemma 5.4.7. \(\square\)

The proof above implies the following useful property of the event \(\mathcal{N}A(\varepsilon, \delta)\).

**Lemma 5.4.8.** Let \(\eta, \varepsilon, \delta > 0\) with \(0 < 10\varepsilon < \delta < 1\). If \(\omega_\eta \in \mathcal{N}A(\varepsilon, \delta)\), then we have \(|\mathcal{C}_1^\eta(\delta)| \leq 32\varepsilon^{-2}\).

**Proof of Lemma 5.4.8.** Let \(C, C' \in \mathcal{C}_1^\eta(\delta)\) be clusters with diameter at least \(\delta\) in the horizontal direction. The proof of Lemma 5.4.7 shows that on the event \(\mathcal{N}A(\varepsilon, \delta)\) \(L(K_\varepsilon(C))\) and \(L(K_\varepsilon(C'))\) are disjoint. The same holds for pairs of clusters with vertical diameter at least \(\delta\). Thus \(|\mathcal{C}_1^\eta(\delta)| \leq 2(2[1/\varepsilon])^2) \leq 32\varepsilon^{-2}\). \(\square\)

### 5.4.1 Bounds on the probability of the events \(\mathcal{N}C(\varepsilon, \delta)\) and \(\mathcal{N}A(\varepsilon, \delta)\)

Our aim in this section is to prove the following bound on the probability of \(\mathcal{E}(\varepsilon, \delta)\).

**Proposition 5.4.9.** Let \(\varepsilon, \delta > 0\) with \(0 < 10\varepsilon < \delta < 1\). Suppose that Assumptions I-III hold. Then there are positive constants \(C = C(\delta), \lambda > 0\) such that for all \(\eta \in (0, \varepsilon)\) we have

\[
\mathbb{P}_\eta(\mathcal{E}(\varepsilon, \delta)^c) \leq C\varepsilon^\lambda.
\]

The proof of the proposition above follows from Lemma 5.4.10 and 5.4.11 below. We start by an upper bound on the probability of \(\mathcal{N}A(\varepsilon, \delta)\).

**Lemma 5.4.10.** Suppose that Assumptions I-III hold. Let \(\varepsilon, \delta > 0\) with \(0 < 10\varepsilon < \delta < 1\). Then there are constants \(C = C(\delta), \lambda > 0\) such that

\[
\mathbb{P}_\eta(\mathcal{N}A(\varepsilon, \delta)^c) \leq C\varepsilon^\lambda
\]

(5.21)

for all \(\eta < \varepsilon\). In particular, \(|\mathcal{C}_1^\eta(\delta)|\) is tight in \(\eta\) for all fixed \(\delta > 0\).

**Proof of Lemma 5.4.10.** For \(\varepsilon, \delta > 0\) with \(0 < 10\varepsilon < \delta < 1\) simple union bounds together with Lemmas 5.3.10 and 5.3.11 give

\[
\mathbb{P}_\eta(\mathcal{N}A_1(\varepsilon, \delta)^c) \leq 10\varepsilon^{-2} \left(\frac{\varepsilon}{\delta}\right)^{2+\lambda_{1,3}} = 10 \frac{\varepsilon^{\lambda_{1,3}}}{\delta^{2+\lambda_{1,3}}},
\]

\[
\mathbb{P}_\eta(\mathcal{N}A_2(\varepsilon, \delta)^c) \leq 40\varepsilon^{-1} \left(\frac{\varepsilon}{\delta}\right)^2 = 40 \frac{\varepsilon}{\delta^{2}}.
\]

This combined with the definition of the event \(\mathcal{N}A(\varepsilon, \delta)\) provides the desired upper bound.

The tightness of \(|\mathcal{C}_1^\eta(\delta)|\) follows from the combination of Lemma 5.4.8 and (5.21). \(\square\)
Lemma 5.4.11. Suppose that Assumptions I-III hold. Let \( \varepsilon, \delta \) with \( 0 < 10\varepsilon < \delta < 1 \). Then there is a constant \( C > 0 \) such that for all \( \eta \in (0, \varepsilon) \) we have
\[
\mathbb{P}_\eta(\mathcal{N}C(\varepsilon, \delta)^c) \leq C \frac{\varepsilon}{\delta^2}.
\]

Proof of Lemma 5.4.11. A simple union bound combined with Lemma 5.3.10 provides the desired bound. \( \square \)

5.5 Construction of the set of large clusters in the scaling limit

Now we are ready to construct the limiting object from Theorem 5.1.1. Before we do so, Corollary 5.3.2 combined with Assumption IV and Lemma 5.3.16 implies that there is a coupling denoted by \( \mathbb{P} \) of \( \omega_\eta \)'s for \( \eta \geq 0 \) such that
\[
\mathbb{P}(\omega_\eta \to \omega_0 \text{ as } \eta \to 0) = 1,
\]
where \( \omega_0 \) has law \( \mathbb{P}_0 \), which we use in the following.

Fix some \( \delta > 0 \). Let \( \omega \in \mathcal{H} \) be a quad-crossing configuration. We define
\[
n_0(\omega) := \inf \left\{ n \geq 0 \mid \omega \in \mathcal{E}(3^{-n'}, \delta) \text{ for all } n' \geq n \right\},
\]
where we use the convention that the infimum of the empty set is \( \infty \). From the construction above, it is clear that the set \( \mathcal{E}(3^{-n}, \delta) \in \text{Borel}(\mathcal{T}_C) \), hence the function \( n_0 \) is \( \text{Borel}(\mathcal{T}_C) \) measurable. Note that \( \omega_\eta \in \mathcal{E}(\eta/10, \delta) \) for \( 0 < \eta < 10\delta \). Hence \( n_0(\omega_\eta) < \infty \). Furthermore, we write \( g_n(\omega, \delta) \) for the number of good subgraphs in \( G_{3-n}(\omega) \).

Let \( \eta > 0 \), \( n \geq n_0(\omega_\eta) \), and \( H_\eta \) be a good subgraph in \( G_{3-n}^\eta = G_{3-n}(\omega_\eta) \). Proposition 5.4.3 shows that for all \( n' \geq n \), there is a unique good subgraph \( H_\eta^{n'} \subseteq G_{3-n'}^\eta \) such that \( U(H_\eta^{n'}) \supseteq U(H_\eta^n) \).

Let \( g_n^\eta = g_n(\omega_\eta, \delta) \). For each \( n \geq 0 \), we fix an ordering of the graphs with vertex sets in \( B_{3-n} \). For \( j = 1, 2, \ldots, g_n^\eta \), let \( H_{j,n_0}^\eta := H_{j,n_0}(\omega_\eta)(\omega_\eta) \) denote the \( j \)th good subgraph of \( G_{3-n_0}^\eta \). Then for \( n \geq n_0(\omega_\eta) \), let \( H_{j,n}^\eta \) denote the unique good subgraph of \( G_{3-n}^\eta \) such that \( U(H_{j,n_0}^\eta) \supseteq U(H_{j,n}^\eta) \).

For \( \eta \geq 0 \) and \( j = 1, 2, \ldots, g_n^\eta \), we set
\[
\mathcal{C}_j^\eta(\delta) := \bigcap_{n \geq n_0(\omega_\eta)} U(H_{j,n}^\eta)
\]
on the event \( n_0(\omega_\eta) < \infty \), while on the event \( n_0(\omega_\eta) = \infty \) we set \( \mathcal{C}_j^\eta(\delta) = \{-1/2, 1/2\} \) for all \( j \geq 1 \). (Note that we can replace \( \{-1/2, 1/2\} \) by any disconnected subset of \( \Lambda_1 \).) Since the sequence of compact sets \( U(H_{j,n}^\eta) \) is decreasing, the intersection in (5.22) is non-empty on the event \( n_0(\omega_\eta) < \infty \). Proposition 5.4.3 shows that for \( \eta > 0 \), we get the collection of clusters \( \mathcal{C}_1^\eta(\delta) \), that is,
\[
\mathcal{C}_1^\eta(\delta) = \{ C_j^\eta(\delta) : 1 \leq j \leq g_n^\eta \}.
\]
Chapter 5. Conformal measure ensembles for percolation and FK-Ising

Before we state and prove the following precise version of Theorem 5.1.1, let us comment on the topology used there. We employ a slightly different topology than the one in (5.5), defined as follows.

Let $\mathcal{C}$ denote the set of non-empty closed subsets of $\Lambda_1$ endowed with the Hausdorff distance $d_H$ as defined in (5.3). Let $l(\mathcal{C})$ denote the space of sequences in $\mathcal{C}$. We endow it with the metric $d_l$ defined as

$$d_l(C, C') := \sum_{j=1}^{\infty} d_H(C_j, C'_j) 2^{-j} \quad (5.23)$$

for $C = (C_j)_{j \geq 1}, C' = (C'_j)_{j \geq 1}$. Note that convergence in $d_l$ is equivalent with coordinate-wise convergence. Furthermore, $l^\infty(\mathcal{C})$ inherits the compactness from $\mathcal{C}$.

For $\eta \geq 0$ we extend the definition (5.22), by setting $C^\eta_1(\delta) := \{-1/2, 1/2\}$ for $j > g_n^\eta$. We write $C^\eta_1(\delta) := (C^\eta_j(\delta))_{j \geq 1}$.

For a quad-crossing configuration $\omega$, $C^\eta_1 = C^\eta_1(\omega)$ denotes the vector of all macroscopic clusters in $\omega$ defined as follows. The first $g_n(\omega, 3^{-1})$ entries of $C^\eta_1(\omega)$ coincide with those of $C^\eta_1(\omega, 3^{-1})$. For $m \geq 4$, the next $g_n(\omega, m^{-1}) - g_n(\omega, (m-1)^{-1})$ entries coincide with those elements in $C^\eta_1(\omega, m^{-1})$ which are unlisted earlier in $C^\eta_1(\omega)$, with their relative ordering.

Now we are ready to state the following precise and slightly stronger version of Theorem 5.1.1.

**Theorem 5.5.1.** Suppose that Assumptions I-IV hold. Let $\delta > 0$ and $\mathbb{P}$ be a coupling where $\omega_\eta \to \omega_0$ a.s. as $\eta \to 0$. Then $C^\eta_1(\delta) \to C^0_1(\delta)$ in probability in the metric $d_l$ as $\eta \to 0$. In particular, the pair $(\omega_\eta, C^\eta_1(\delta))$ converges in distribution to $(\omega_0, C^0_1(\delta))$ as $\eta \to 0$. Moreover, same convergence result holds for $C^\eta_1$. Furthermore, $C^\eta_1(\delta)$ and $C^0_1$ are measurable functions of $\omega_0$.

**Remark 5.5.2.** Note that the connected sets of $\Lambda_1$ form a compact subspace of $\mathcal{C}$. Hence $\{-1/2, 1/2\}$ is separated from the clusters $C^\eta_j$ for $j = 1, \ldots, g_n^\eta$. Thus the convergence of the vectors $C^\eta_1(\delta)$ in the metric $d_l$ implies the convergence of $C^\eta_1(\delta)$ in the topology (5.5). Namely, the bijection is given by the ordering of the entries in the corresponding vectors, while the proof of Lemma 5.4.8 implies that, in the sequence, there is no pair of clusters converging to the same closed set. The convergence in the metric (5.6) follows from the equivalence of the metrics $d_H$ and $D_H$.

Before we turn to the proof of Theorem 5.5.1, we prove the following lemma.

**Lemma 5.5.3.** Suppose that Assumptions I-IV hold. Let $\mathbb{P}$ be a coupling such that $\omega_\eta \to \omega_0$ $\mathbb{P}$-a.s. as $\eta \to 0$. Then

$$\mathbb{P}(n_0(\omega_0) = \infty) = 0.$$

Moreover, $n_0(\omega_\eta) \to n_0(\omega_0)$ in probability under $\mathbb{P}$ as $\eta \to 0$.

**Proof of Lemma 5.5.3.** For each fixed $\varepsilon, \delta > 0$ the event $\mathcal{E}(\varepsilon, \delta)$ can be written as a finite union of intersections of some events appearing in Lemma 5.3.18. Thus

$$\mathbb{P}_0(\mathcal{E}(\varepsilon, \delta)^c) = \lim_{\eta \to 0} \mathbb{P}_\eta(\mathcal{E}(\varepsilon, \delta)^c) \leq C\varepsilon^\lambda$$
where $C, \lambda$ as in Proposition 5.4.9. Hence

\[
\sum_{n=1}^{\infty} \mathbb{P}_0(\mathcal{E}(3^{-n}, \delta)^c) < \infty.
\]

Thus the Borel-Cantelli lemma shows that $\mathbb{P}(n_0(\omega_0) = \infty) = 0$.

Let $k \geq 1$. Lemma 5.3.18 and Proposition 5.4.9 implies that

\[
\mathbb{P}(|n_0(\omega_\eta) - n_0(\omega_0)| \geq 1) \\
\leq \mathbb{P}(n_0(\omega_\eta) > k) + \mathbb{P}(n_0(\omega_0) > k) \\
+ \mathbb{P}(|n_0(\omega_\eta) - n_0(\omega_0)| \geq 1, n_0(\omega_0) \vee n_0(\omega_\eta) \leq k) \\
\leq \sum_{l \geq k+1} (\mathbb{P}(\omega_\eta) \mathbb{P}(\mathcal{E}(3^{-l}, \delta)^c) + \mathbb{P}(\mathcal{E}(3^{-l}, \delta)^c)) \\
+ \mathbb{P}(\exists l \leq k \text{ s.t } 1(\omega_\eta \in \mathcal{E}(3^{-l}, \delta)) \neq 1(\omega_0 \in \mathcal{E}(3^{-l}, \delta)) \\
\leq C \sum_{l \geq k+1} 3^{-\lambda l} + \sum_{l=1}^{k} \mathbb{P}(1(\omega_\eta \in \mathcal{E}(3^{-l}, \delta)) \neq 1(\omega_0 \in \mathcal{E}(3^{-l}, \delta)))
\]

with some constant $C > 0$. Taking $\eta \to 0$ in (5.24) with a suitable constant $C'$ we get

\[
\lim_{\eta \to 0} \mathbb{P}(|n_0(\omega_\eta) - n_0(\omega_0)| \geq 1) \leq C' 3^{-\lambda k}
\]

for all $k > 0$. This shows that $n_0(\omega_\eta) \to n_0(\omega_0)$ in probability as $\eta \to 0$, and concludes the proof of Lemma 5.5.3.

\[\square\]

Proof of Theorem 5.5.1. Let $\delta > 0$, and $\mathbb{P}$ be a coupling such that $\omega_\eta \to \omega_0$ a.s. We will work under $\mathbb{P}$ in the following. Note that for each $n \in \mathbb{N}$, the event $\mathcal{E}(3^{-n}, \delta)$, the graph $G_{3-n}(\omega)$ and the good subgraphs of $G_{3-n}(\omega)$ are functions of the outcomes of finitely many arm events appearing in Lemma 5.3.18. Thus each of

- $1\{\omega_\eta \in \mathcal{E}(3^{-n}, \delta)\}$,
- $G_{3,n}(\omega_\eta)$, and
- the ordered set of good subgraphs of $G_{3,n}(\omega_\eta)$

converge to the same quantities with $\omega_\eta$ replaced by $\omega_0$ in probability as $\eta \to 0$. This has the following consequences:

1) with Lemma 5.5.3, we have $n_0(\omega_\eta) \to n_0(\omega_0) < \infty$,

2) $g^\eta_n \to g^0_n$ for all $n \geq 1$, in particular, $g^\eta_{n_0(\omega_\eta)} \to g^0_{n_0(\omega_0)}$,

3) $H^n_{j,n} \to H^0_{j,n}$ for $j = 1, 2, \ldots, g_{n_0(\omega_\eta)}$ and $n \geq n_0(\omega_\eta)$

in probability as $\eta \to 0$. Let $n \geq n_0(\omega_\eta) \vee n_0(\omega_0)$, then

\[
d_H(C^\eta_j, C^0_j) \leq d_H(C^\eta_j, U(H^\eta_{j,n})) + d_H(U(H^\eta_{j,n}), U(H^0_{j,n})) + d_H(U(H^0_{j,n}), C^0_j) \\
\leq 3^{-n} + d_H(U(H^\eta_{j,n}), U(H^0_{j,n})) + 3^{-n}
\]

(5.25)
Figure 5.2: Illustration of a cluster in $D$. The small open circles denote the interior of the loop $l$. The shaded area intersected with the cluster of the loop is equal to $B(\mathcal{E})$.

for $j = 1, 2, \ldots, g_{n_0} \land g_{n_0}$. Thus taking the limit $\eta \to 0$ in (5.25), by 1)-3) above, we get

$$\lim_{\eta \to 0} \mathbb{P}(d_H(C_\eta^j, C_0^j) > 3 \cdot 3^{-n}, n \geq n_0(\omega_0) \lor n_0(\omega_\eta)) = 0$$

(5.26)

for $j \geq 1$. Then taking the limit $n \to \infty$, Lemma 5.5.3 shows that $C_\eta^j \to C_0^j$ in Hausdorff metric in probability as $\eta \to 0$ for all $j \geq 1$. Since convergence in $l^\infty(\mathcal{C})$ coincides with coordinate-wise convergence, we get that $\lim_{\eta \to 0} C_\eta^1(\delta) = C_0^1(\delta)$ in probability, as required.

The proof of the claims of Theorem 5.5.1 for $C_\eta^1$ is analogous. It follows from the convergence of $C_\eta^1(\delta)$ with $\delta = 3^{-m}$ for $m \geq 1$. The measurability of $C_0^1(\delta)$ and $C_0^0$ with respect to $\omega_0$ follows easily from their definition involving arm events (see Remark 5.3.6). Thus the proof of Theorem 5.5.1 is complete.

5.6 Scaling limits in a bounded domain

In this section we will deduce the convergence of all clusters and "pieces" of clusters contained in a bounded domain $D$ from the convergence of clusters and loops completely contained in $\Lambda_k \supset D$, for some $k$ sufficiently large. We denote $B_D^\eta(\delta)$ the collection of all clusters or portions of clusters of diameter at least $\delta$ contained in $D^\eta$, where $D^\eta$ denotes an appropriate discretization of $D$. In the case of $\mathbb{Z}^2$, the
boundary of $D^n$ is a circuit in the medial lattice that surrounds all the vertices of $\mathbb{Z}^2$ contained in $D$ and minimizes the distance to $\partial D$. Analogously, in the case of the triangular lattice, $\mathbb{T}$, the boundary of $D^n$ is a circuit in the dual (hexagonal) lattice that surrounds all the vertices of $\mathbb{T}$ contained in $D$ and minimizes the distance to $\partial D$.

More precisely, for every cluster $\mathcal{C} \in \mathcal{C}^n(\delta)$ that intersect $D^n$, consider the set of all connected components $\mathcal{B}$ of $\mathcal{C} \cap D^n$ with diameter at least $\delta > 0$. For every $\eta, \delta > 0$, we let $\mathcal{B}_D^n(\delta)$ denote the union of $\mathcal{C}^n(\delta)$ with the set of all such connected components $\mathcal{B}$. (Note that clusters contained in $\Lambda_k$ but not completely contained in $D^n$ are split into different elements of $\mathcal{B}_D^n(\delta)$. See figure 5.2.) For the case of Bernoulli percolation, the collection $\mathcal{B}_D^n(\delta)$ is precisely the set of all clusters in $D^n$ with closed boundary condition.

As in Section 5.5, instead of the collection $\mathcal{B}_D^n(\delta)$, we consider the sequence $\mathcal{B}_D^n(\delta)$ of clusters with diameter at least $\delta$, with the metric $d$. Now we are ready to state the theorem on the convergence of all portions of clusters in $\sigma_\eta \cap D$ for a bounded domain $D$.

**Theorem 5.6.1.** Suppose that Assumptions I-IV hold. Let $D$ be a simply connected bounded domain with piecewise smooth boundary. Let $\mathbb{P}$ be a coupling where $(\omega_\eta, L_\eta) \rightarrow (\omega_0, L_0)$ a.s. as $\eta \rightarrow 0$. Then, for any $\delta > 0$, $\mathcal{B}_D^n(\delta) \rightarrow \mathcal{B}_D^0(\delta)$ in probability in the metric $d_i$ as $\eta \rightarrow 0$. In particular, the triple $(\omega_\eta, L_\eta, \mathcal{B}_D^n(\delta))$ converges in distribution to $(\omega_0, L_0, E^n_D(\delta))$ as $\eta \rightarrow 0$. Moreover, the same convergence result holds for $\mathcal{B}_D^{0}$. Furthermore, $\mathcal{B}_D^n(\delta)$ and $\mathcal{B}_D^0$ are measurable functions of the pair $(\omega_0, L_0)$.

**Proof.** Let $(\omega_\eta, L_\eta)$ and $(\omega_0, L_0)$ be as in the statement of Theorem 5.6.1. The probability that all the clusters that intersect $D$ are completely contained in $\Lambda_k$ is at least one minus the probability of having a red arm from the boundary of $D$ to $\partial \Lambda_k$. The latter probability goes to zero as $k \rightarrow \infty$, hence there is a finite $k \in \mathbb{N}$ such that there is no red arm from $D$ to $\partial \Lambda_{k-1}$ in $\omega_0$. We take the smallest such $k$. With this choice, all clusters in $\mathcal{C}^n$ that intersect $D$ are contained in $\Lambda_k$.

We first give an orientation to the loops contained in $\Lambda_k$ in such a way that clockwise loops are the outer boundaries of red clusters and counterclockwise loops are the outer boundaries of blue clusters. For each clockwise loop $\ell$ intersecting $\partial D$, we consider all excursions $\mathcal{E}$ inside $D$ of diameter at least $\delta$. Each excursion $\mathcal{E}$ runs from a point $s_{in}$ on $\partial D$ to a point $s_{out}$ on $\partial D$. We call the counterclockwise segment of $\partial D$ from $s_{in}$ to $s_{out}$ the base of $\mathcal{E}$. We call $\mathcal{E}$ the concatenation of $\mathcal{E}$ with its base. We define the interior $I(\mathcal{E})$ of $\mathcal{E}$ to be the closure of the set of points with nonzero winding number for the curve $\mathcal{E}$.

We call $\mathcal{E}_{\ell}$ the collection of all clockwise excursions in $D$ of the same loop $\ell$ with base contained inside the base of $\mathcal{E}$. If $\mathcal{C}$ is the cluster whose outer boundary is the loop $\ell$, we define $\mathcal{B}(\mathcal{E})$ as follows:

$$\mathcal{B}(\mathcal{E}) := \overline{I(\mathcal{E}) \setminus \{ \cup_{\mathcal{E}' \in \mathcal{E}_{\ell}} I(\mathcal{E}') \}} \cap \mathcal{C},$$

where by $\cup_{\mathcal{E}' \in \mathcal{E}_{\ell}} I(\mathcal{E}')$ we mean $\lim_{\xi \rightarrow 0} \cup_{\mathcal{E}' \in \mathcal{E}_{\ell}, \text{diam}\mathcal{E}' > \xi} I(\mathcal{E}')$, and the limit exists because it is the limit of an increasing sequence of closed sets.

For any $\delta > 0$, $\mathcal{B}_D^n(\delta)$ is the collection of all sets $\mathcal{B}(\mathcal{E})$ defined above, for all clockwise excursions $\mathcal{E}$ in $D$ of diameter at least $\delta$. 

For any $\eta > 0$, the collection $B_D^n(\delta)$ contains all clusters completely contained in $D$ plus all the connected components of the intersections of clusters in $\Lambda_k$ with $D$. $B_D^n(\delta)$ can be obtained with the following construction which mimics the continuum construction given earlier. We first give an orientation to the loops contained in $\Lambda_k$ in such a way that loops that have red in their immediate interior are oriented clockwise and loops that have blue in their immediate interior are oriented counterclockwise. For each clockwise loop $\ell^n$ intersecting $\partial D^n$, we consider all excursions $E^n$ inside $D^n$ of diameter at least $\delta$. Each excursion $E^n$ runs from a point $s_{\text{in}}^n$ on $\partial D^n$ to a point $s_{\text{out}}^n$ on $\partial D^n$. We call the counterclockwise segment of $\partial D^n$ from $s_{\text{in}}^n$ to $s_{\text{out}}^n$ the base of $E^n$. We call $\overline{E}$ the concatenation of $E^n$ with its base. We define the interior $I(\overline{E})$ of $\overline{E}$ to be the set of hexagons contained inside $\overline{E}$.

We call $E_n$ the collection of all clockwise excursions in $D^n$ of the same loop $\ell^n$ with base contained inside the base of $E^n$. If $C^n$ is the cluster whose outer boundary is the loop $\ell^n$, we define $B^n(E^n)$ as follows:

$$B^n(E^n) := I(\overline{E}) \setminus \left\{ \bigcup_{(E^n)^n} \{ (E^n)^n \} \right\} \cap C^n.$$

We now note that the almost sure convergence $(\omega_n, L_n) \rightarrow (\omega_0, L_0)$, combined with Lemma 5.3.10, implies the same for the excursions in $D$. (Lemma 5.3.10 insures, via standard arguments, that an excursion cannot come close to the boundary of $D$ without touching it, so that large lattice and continuum excursions will match exactly for $\eta$ sufficiently small. For more details on how to use Lemma 5.3.10, the interested reader is referred to Lemma 6.1 of [22].) Together with the convergence of the clusters, this implies that $(\omega_n, L_n, B^n_D(\delta))$ converges in distribution to $(\omega_0, L_0, B_D(\delta))$ as $n \rightarrow 0$. The above result is valid for any $\delta > 0$, so letting $\delta \rightarrow 0$ gives the second part of the theorem.

### 5.7 Limits of counting measures of clusters

Herein we state and prove Theorem 5.7.2, a precise and slightly stronger version of Theorem 5.1.2. We do this for the more general case of (portions of) clusters $B_D^n(\delta)$ in a domain with piecewise smooth boundary $D$. The convergence of measures of the clusters which are completely contained in $\Lambda_k$ follows immediately. For ease of notation we assume $D$ to be $\Lambda_1$.

Let $\mathfrak{M}$ denote the set of finite Borel measures on $\Lambda_1$ endowed with the Prokhorov metric. Recall that $\mathfrak{M}$ is a separable metric space.

For $\delta, \varepsilon > 0$, $\eta > 0$ and $S \subseteq \Lambda_1$ we define

$$\mu^n_{S,n} := \sum_{z \in \mathbb{Z}[i]: A_{z;3^{-n}/2}(3^{-n}z) \cap S \neq \emptyset} \mu^n_{1,A(3^{-n}z;3^{-n}/2,\delta/2,3^{-n})}, \tag{5.27}$$

That is the sum of counting measures $\mu^n_{1,A(z;3^{-n}/2,\delta)}$ where the inner box $A_{z;3^{-n}/2}(z)$ self or one of its neighbours has nonempty intersection with $S$.

Simple arguments show the following:

**Observation 5.7.1.** Let $B$ be a Borel set of $\mathbb{C}$ and $S \subseteq \Lambda_1$. Then $\mu^n_{S,n}(B) \geq \mu^n_{S,n'}(B)$ for $n' \geq n$ with probability 1 for fixed $\eta > 0$. 


It is easy to check that for all fixed $\eta > 0$ and $B \in \mathcal{B}_{\Lambda_1}(\delta)$ the following limit exists
\[
\lim_{n \to \infty} \mu^n_{B,n}
\] (5.28)
and is actually equal to $\mu^n_B$ as defined in (5.7).

This motivates us to define, for any cluster $B \in \mathcal{B}_{\Lambda_1}(\delta)$, $\mu^n_B$ by (5.28) with $\eta = 0$ if the limit exists, and set $\mu^n_B = 0$ when it does not.

Let $l(\mathcal{M})$ denote the set of infinite sequences in $\mathcal{M}$ with bounded distance from the empty measure. Similarly to (5.23), we set
\[
d_i(\nu, \phi) := \sum_{j=1}^{\infty} \frac{d_P(\nu_j, \phi_j)}{1 + d_P(\nu_j, \phi_j)} 2^{-j}
\]
for $\nu, \phi \in l(\mathcal{M})$. It is easy to check that $l(\mathcal{M})$ is separable, but not compact. Let $h^0(\delta) := |\mathcal{B}_{\Lambda_1}(\delta)|$, for $\eta \geq 0$. It follows from Lemma 5.5.3, together with the tightness of the number of excursions in $\Lambda_1$, of diameter at least $\delta$, that $h^0(\delta)$ is a.s. finite. For $\eta \geq 0$ we define $\mu^n = (\mu^*_j)_{j \geq 1}$, the vector of these measures, where $\mu^*_j := (\mu^n_j)$ is as above for $j = 1, 2, \ldots, h^0(\delta)$, and we set $\mu^*_j = 0$ for $j > h^0(\delta)$. We define $\mu^\eta$ similarly to $\mathcal{L}^\eta$.

Now we are ready to state the main result from this section.

**Theorem 5.7.2.** Suppose that Assumptions I-IV hold. Let $D$ be a simply connected bounded domain with piece-wise smooth boundary. Let $\mathbb{P}$ be a coupling where $(\omega, L) \to (\omega_0, L_0)$ a.s. as $\eta \to 0$. Then $\mu^n_D(\delta) \to \mu^n_D(\delta)$ in probability as $\eta \to 0$, where $\mu^n_D(\delta)$ is a measurable function of the pair $(\omega_0, L_0)$. In particular, the triple $(\omega, L, \mu^n_D(\delta))$ converges in distribution to $(\omega_0, L_0, \mu^n_D(\delta))$ as $\eta \to 0$. The same convergence result holds when $\mu^n_D(\delta)$ is replaced by $\mu^n_D$.

The same conclusion holds for the measures of the clusters in $\check{\mathcal{C}}$ which intersect a bounded domain $D$, that is, we keep the information of connections outside $D$.

**Remark 5.7.3.** Lemma 5.8.2 shows that clusters whose diameter is at least $\delta > 0$ have nonzero mass. Thus the convergence in Theorem 5.7.2 implies convergence in the metric (5.5) and so Theorem 5.1.2 is proved.

Let us first show that Theorem 5.1.3 follows easily from Theorems 5.5.1 and 5.7.2.

**Proof of Theorem 5.1.3.** The proof is analogous to the proof of Theorem 6 of [22], so we only give a sketch. Let $D$ be any bounded subset of $\mathbb{C}$ and $k_1 > k_2$ be such that $D \subset \Lambda_{k_2}$. The measures $\mathbb{P}_{k_1}$ and $\mathbb{P}_{k_2}$ can be coupled in such a way that they coincide inside $D$, in the sense that they induced the same marginal distribution on $(\mathcal{C}_D, \mathcal{M}_D)$. This is because they are obtained from the scaling limit of the same full-plane lattice measure $\mathbb{P}_\eta$. The consistency relations needed to apply Kolmogorov’s extension theorem are then satisfied, which insures the existence of a limit $\mathbb{P}$. 

The following lemma plays an important role in the proof of Theorem 5.7.2. Let $||\nu||_{TV}$ denote the total variation of a signed measure $\nu$. 

Lemma 5.7.4. Suppose that Assumptions I-III hold. Let \( \delta > 0 \). Then there are positive constants \( C = C(\delta), \varphi \) such that, for \( n \in \mathbb{N} \) and \( \eta > 0 \) with \( 0 < 10\eta < 3^{-n} < \delta/10 \)

\[
\mathbb{P}_\eta(\exists B \in \mathcal{B}_{\Lambda_1}(\delta), S \subseteq \Lambda_1 \text{ s.t.: } d_H(B, S) < \varepsilon/2, ||\mu_B^n - \mu_{S,n}^\eta||_{TV} \geq \varepsilon/\varphi) \leq C \cdot \varepsilon \varphi
\]

where \( \varepsilon = 3^{-n} \).

Proof of Theorem 5.7.2 given Lemma 5.7.4. Let \( \mathbb{P} \) as in Theorem 5.7.2, \( \delta > 0 \). It follows from Theorem 5.6.1 that the clusters in \( \mathcal{B}_{\Lambda_1}(\delta) \) converge in probability as \( \eta \to 0 \).

Moreover, Theorem 5.3.19 shows that each of the measures

\[
\mu_{1, A(3^{-n}, 3^{-n}/2, \delta - 3^{-n})}^\eta \text{ for } n \geq 1 \text{ and } z \in \mathbb{Z}[i] \text{ with } 3^{-n}z \in \Lambda_1.
\]

converge in the Prokhorov metric in probability as \( \eta \to 0 \) to the version of measures where \( \eta \) is replaced by \( 0 \).

This implies that, for all fixed \( n \) and \( S \subseteq \Lambda_1 \), \( \mu_{S,n}^\eta \to \mu_S^0 \) weakly in probability as \( \eta \to 0 \). The monotonicity of the measures \( \mu_{S,n}^\eta \) in \( n \) for a fixed subset \( S \) and fixed \( \eta \) of Observation 5.7.1 carries through the limit as \( \eta \to 0 \), thus the weak limit

\[
\mu_S^0 = \lim_{n \to \infty} \mu_{S,n}^0 \text{ a.s. exists. Furthermore, since each of the measures } \mu_S^0 \text{ is a function of } (\omega_0, L_0) \text{ and a.s. finite, we derive that } \mu_S^0 \text{ is a.s. finite and is a function of } (\omega_0, L_0).
\]

Recall the sequence \( \mathcal{B}_{\Lambda_1}^0(\delta) \) of clusters. Let \( B \) be the \( j \)-th element of this sequence and let \( \mathcal{B}_j^0 \) be the \( j \)-th element of \( \mathcal{B}_{\Lambda_1}^0(\delta) \). Let \( \kappa > 0 \) fixed. Lemma 5.7.4 implies that for some constants \( \varphi, C = C(\delta) \) for \( \kappa > \varepsilon \varphi, \eta < \varepsilon/10 \) and \( 3^{-n} = \varepsilon \) we have

\[
\mathbb{P}(d_P(\mu_{B_j}^0, \mu_{B_j}^\eta) > 3\kappa) \\
\leq \mathbb{P}(d_P(\mu_{B_j}^0, \mu_{B_j}^0) > \kappa) + \mathbb{P}(d_P(\mu_{B_j}^0, \mu_{B_j}^\eta) > \kappa) \\
+ \mathbb{P}(||\mu_{B_j,n}^\eta - \mu_{B_j}^\eta||_{TV} > \kappa, d_H(B, B_j^0) < \varepsilon/2) + \mathbb{P}(d_H(B, B_j^0) \geq \varepsilon/2) \\
\leq \mathbb{P}(d_P(\mu_{B_j}^0, \mu_{B_j}^0) > \kappa) + \mathbb{P}(d_P(\mu_{B_j}^0, \mu_{B_j}^\eta) > \kappa) \\
+ C \kappa + \mathbb{P}(d_H(B, B_j^0) \geq \varepsilon/2)
\] (5.29)

where \( d_P \) denotes the Prokhorov distance of Borel measures.

Now we take the limit first as \( \eta \to 0 \) then as \( n \to \infty \) in (5.29). From the arguments above and Theorem 5.6.1 we deduce that

\[
\lim_{\eta \to 0} \mathbb{P}(d_P(\mu_{B_j}^0, \mu_{B_j}^\eta) > 3\kappa) \leq C \kappa
\]

for all \( \kappa > 0 \). Thus the measures \( \mu_{B_j}^\eta \) tend to \( \mu_{B_j}^0 \) weakly in probability as \( \eta \to 0 \).

Recall that the convergence in \( l^\infty(\mathbb{R}) \) is equivalent with coordinate-wise convergence. Thus \( \mu^\eta(\delta) \to \mu^0(\delta) \) in probability as \( \eta \to 0 \). We have already proved in the lines above that \( \mu^0(\delta) \) is a measurable function of \( (\omega_0, L_0) \), thus we deduced the results in Theorem 5.7.2 for \( \mu^\eta(\delta) \).

The results for \( \mu^\eta \) follow from the lines above by arguments similar to those at the end of the proof of Theorem 5.5.1. This concludes the proof of Theorem 5.7.2. \( \Box \)
We finish this section by proving Lemma 5.7.4 above. Its proof relies on Lemma 5.3.14.

**Proof of Lemma 5.7.4.** Let \( \eta, n, \delta \) as in Lemma 5.7.4. To ease the notation, we set \( \varepsilon := 3^{-n} \), \( \delta' := \delta/2 - 3\varepsilon \) and \( \beta := \frac{\lambda}{2(\lambda + \lambda_1)} \), with \( \lambda_1 \) as in Lemma 5.3.9 while \( \lambda \) as in Lemma 5.3.12.

Let \( \nu^\eta_{\varepsilon, \beta} \) denote the normalized counting measure of the vertices close to the boundary of \( \Lambda_1 \) which have an open arm to distance \( 5\varepsilon^\beta \). That is,

\[
\nu^\eta_{\varepsilon, \beta} := \frac{\eta^2}{\pi_1(\eta, 1)} \sum_{v \in A(0;1-\varepsilon, 1) \cap \eta V} \delta_0(\varepsilon) \overline{\partial \Lambda_{5\varepsilon^\beta}(v)}.
\]

Furthermore, we define the following collection of ‘pivotal’ boxes:

\[
\text{Piv}^\eta(\varepsilon, \varepsilon^\beta) := \{ \Lambda_{\varepsilon/2}(\varepsilon z) \mid z \in \mathbb{Z}[i] \cap \Lambda_{\varepsilon-1+1}; \omega_{\eta} \in A_{(1010), \delta}(\varepsilon z; 3\varepsilon/2, \varepsilon^\beta) \}.
\]

Let \( \mathcal{B} \in \mathcal{R}^n_{\Lambda_1}(\delta) \) and \( S \subseteq \Lambda_1 \) such that \( d_H(S, \mathcal{B}) < \varepsilon/2 \). Note that \( d_H(S, \mathcal{B}) < \varepsilon/2 \) implies that the counting measure \( \mu^\eta_{S,n} \) is larger or equal the counting measure \( \mu^\eta_{\mathcal{B}} \). As a consequence it is easy to check that, for these \( \mathcal{B} \) and \( S \), we have

\[
\| \mu^\eta_{S,n} - \mu^\eta_{\mathcal{B}} \|_{TV} \leq \| \mu^\eta_{\varepsilon, \beta} \|_{TV} + \sum_{z \in \mathbb{Z}[i] \cap \Lambda_{\varepsilon-1+1}} \sup_{\omega_{\eta} \in \mathcal{A}_{(1010), \delta}(\varepsilon z; 3\varepsilon/2, \varepsilon^\beta)} \| \mu^\eta_{1, \mathcal{A}(\varepsilon z; 3\varepsilon/2, \varepsilon^\beta)} \|_{TV}.
\]

Let \( \varphi > 0 \) to be fixed later and \( a^\eta_\varepsilon := \varepsilon^{-(2+\varphi)} \pi_4^{\eta}(3\varepsilon/2, \varepsilon^\beta) \). From (5.31) we deduce that

\[
\mathbb{P}_\eta(\exists \mathcal{B} \in \mathcal{R}^n_{\Lambda_1}(\delta), S \subseteq \Lambda_1 \text{ s.t. } d_H(S, \mathcal{B}) < \varepsilon/2, \| \mu^\eta_{\mathcal{B}} - \mu^\eta_{S,n} \|_{TV} \geq \varepsilon^\varphi) \\
\leq \mathbb{P}_\eta(\| \mu^\eta_{\varepsilon, \beta} \|_{TV} \geq \frac{1}{2} \varepsilon^\varphi) + \mathbb{P}_\eta(\| \text{Piv}^\eta(\varepsilon, \varepsilon^\beta) \|_{TV} \geq a^\eta_\varepsilon) \\
+ \mathbb{P}_\eta(\sup_{z \in \mathbb{Z}[i] \cap \Lambda_{\varepsilon-1+1}} \| \mu^\eta_{1, \mathcal{A}(\varepsilon z; 3\varepsilon/2, 3\varepsilon)} \|_{TV} > \varepsilon^\varphi/2a^\eta_\varepsilon).
\]

By the Markov inequality, we have

\[
\mathbb{P}_\eta(\| \text{Piv}^\eta(\varepsilon, \varepsilon^\beta) \|_{TV} \geq a^\eta_\varepsilon) \leq C_1 \varepsilon^\varphi
\]

for some positive constant \( C_1 = C_1(\delta) \) for all \( \varphi > 0 \).

Now we bound the third term in (5.32). With some positive constants \( C_2, C_3, C_4 \)
depending on \( \delta \) we have

\[
\mathbb{P}_\eta ( \sup_{z \in \Lambda_{\varepsilon^{-1}+1} \cap \mathbb{Z}[i]} ||\mu_{1,A(\varepsilon z; 3\varepsilon/2, 3\varepsilon)} ||_{TV} > \varepsilon^\phi/2 a_\varepsilon^\eta ) \\
\leq C_2 \varepsilon^{-2} \mathbb{P}_\eta ( ||\mu_{1,A(3\varepsilon/2, 3\varepsilon)} ||_{TV} > \varepsilon^\phi/2 a_\varepsilon^\eta ) \\
= C_2 \varepsilon^{-2} \mathbb{P}_\eta ( |Y_{3\varepsilon}| \geq \varepsilon^\phi \eta^{-2} \pi_{1}(\eta, 1)/2 a_\varepsilon^\eta ) \\
\leq C_2 \varepsilon^{-2} \exp \left( -C_3 \varepsilon^2 \frac{\pi_{1}(\eta, 1)}{\pi_{1}(\eta, 3\varepsilon) \pi_{1}(3\varepsilon/2, \varepsilon^\beta) } \right) \\
\leq C_2 \varepsilon^{-2} \exp \left( -C_4 \varepsilon^2 \frac{\pi_{1}(3\varepsilon, \varepsilon^\beta)}{\pi_{1}(3\varepsilon/2, \varepsilon^\beta)} \pi_{1}(\varepsilon^\beta, 1) \right),
\]

where in the second inequality we used Lemma 5.3.14 and in the last line we used Lemma 5.3.8 twice. Lemmas 5.3.9 and 5.3.12, (5.34) and the choice of \( \beta \) give that

\[
\mathbb{P}_\eta ( \sup_{z \in \Lambda_{\varepsilon^{-1}+1} \cap \mathbb{Z}[i]} ||\mu_{1,A(\varepsilon z; 3\varepsilon/2, 3\varepsilon)} ||_{TV} > \varepsilon^\phi/2 a_\varepsilon^\eta ) \leq C_2 \varepsilon^{-2} \exp( -C_5 \varepsilon^2 \phi + \lambda (\beta - 1) + \lambda_1 \beta ) \\
= C_2 \varepsilon^{-2} \exp( -C_5 \varepsilon^2 \phi - \lambda/2 )
\]

with \( C_5 > 0 \). Computations similar to those above give the following upper bound for the second term in (5.32):

\[
\mathbb{P}_\eta ( ||\mu_{\varepsilon^\beta} ||_{TV} \geq \frac{1}{2} \varepsilon^\phi ) \leq C_6 \varepsilon^{-\beta} \exp \left( -C_7 \varepsilon^{\phi - \beta} \frac{\pi_{1}(\eta, 1)}{\pi_{1}(\eta, 3\varepsilon) \pi_{1}(3\varepsilon/2, \varepsilon^\beta)} \right) \\
\leq C_6 \varepsilon^{-\beta} \exp \left( -C_8 \varepsilon^{\phi - \beta + \lambda_1} \right)
\]

for suitable constants \( C_6, C_7, C_8 \). We set \( \varphi = \frac{\lambda (\beta (1 - \lambda_1))}{4} > 0 \). A combination of (5.32), (5.33), (5.35) and (5.36) finishes the proof of Lemma 5.7.4.

\[\square\]

### 5.8 Properties of the continuum clusters and their normalized counting measures

We start with the connections between the clusters and their counting measures. The first result of the section shows, roughly speaking, that the scaling limit of the clusters as closed sets contains the same information as their normalized counting measures. Then we show conformal invariance of the clusters and conformal covariance of their normalized counting measures.

#### 5.8.1 Basic properties

Recall the notation \( \mathcal{C}_\eta(\delta) \) from (5.2). We set \( \mathcal{C}^0 = \bigcup_{n=1}^{\infty} \mathcal{C}^0(3^{-n}) \). For \( \mathcal{C} \in \mathcal{C}^0 \) and \( \psi > 0 \) we write

\[
\tilde{\mu}_{\mathcal{C}, \psi}^0 := \frac{4 \psi^2}{\pi_{1}(2\psi, 1)} \sum_{z \in \mathbb{Z}[i]: \Lambda_{\psi/2}(\psi z) \cap \mathcal{C} \neq \emptyset} \delta_{\psi z}.
\]
Theorem 5.8.1. Suppose that Assumptions I-IV hold. Then \( \text{supp}(\mu^0_C) = C \) for all \( C \in \mathcal{C}^0 \). Moreover,

\[
\tilde{\mu}^0_{C,\psi} \to \mu^0_C \text{ weakly in probability as } \psi \to 0
\]

for all \( C \in \mathcal{C}^0 \).

The proof of the theorem above relies on the following two lemmas.

Lemma 5.8.2. Assume that Assumptions I-III hold. Let \( k, \delta > 0 \). Then for all \( \varphi > 0 \) there is \( x_\varphi = x_\varphi(k, \delta) > 0 \) so that

\[
\mathbb{P}_\eta(\exists C \in \mathcal{R}_k^\eta(\delta) \text{ with } ||\mu^0_C||_{TV} < x_\varphi) < \varphi
\]

for all \( \eta \in (0, \delta) \).

Proof of Lemma 5.8.2. For critical percolation the proof of Lemma 5.8.2 follows from the proof of Theorem 3.1.2: (3.34) with \( x = 0 \) can be shown in the same manner as for \( x > 0 \). Alternatively, Lemma 5.8.2 can be deduced from a combination of [17, Lemma 4.4 and part i) of Theorem 3.1 and 3.3].

It is easy to verify that actually all these arguments just need Assumptions I-III.

The second is essentially [38, Proposition 4.13] see also [38, Eqn. (4.39)]. Let \( A \) be the annulus \( A = A(a, b) \) with \( 0 < a < b \) and \( C \in \mathcal{C}^0 \). For \( \eta \geq 0 \) and \( \psi > 0 \) we set

\[
\tilde{\mu}^\eta_{A,\psi} := \frac{4\psi^2}{\pi_1^\eta(2\psi, 1)} \sum_{z \in \mathbb{Z}[i] \cap \Lambda_{\psi/1}} 1\{\Lambda_{\psi/2}(\psi z) \leftrightarrow \partial \Lambda_b\} \delta_{\psi z}.
\]

Lemma 5.8.3 (Proposition 4.13 of [38]). Suppose that Assumptions I-IV hold. Let \( f : \mathbb{C} \to \mathbb{R} \) be a continuous function with compact support, and \( A = A(a, b) \) an annulus with \( 0 < a < b \). Then

\[
\tilde{\mu}^0_{A,\psi}(f) \to \mu^0_A(f) \text{ in } L^2 \text{ as } \psi \to 0.
\]

Remark 5.8.4. For the proof of Theorem 5.8.1 convergence in probability is enough in (5.40).

Proof of Theorem 5.8.1. Since \( \mathcal{C}^0 = \bigcup_{n=1}^{\infty} \mathcal{C}^0(3^{-n}) \) and \( \mathcal{C}^0(3^{-n}) = \bigcup_{k \in \mathbb{N}} \mathcal{C}^0_k(3^{-n}) \), it is enough to show the required equalities hold with probability 1 for all \( C \in \mathcal{C}^0_k(\delta) \) for any fixed \( \delta > 0 \) and \( k \in \mathbb{N} \). We will work under a coupling \( \mathbb{P} \) such that \( \omega_\eta \to \omega_0 \) a.s.

The proofs of Theorems 5.5.1 and 5.7.2 show that \( \text{supp}(\mu^0_C) \subseteq C \) for all \( C \in \mathcal{C}^0(\delta) \) with probability 1. We turn to the proof of \( \text{supp}(\mu^0_C) \supseteq C \). Let \( \varphi > 0 \) and \( x_\varphi \) as in Lemma 5.8.2. By covering \( \Lambda_k \) with at most \( 4(k/\varepsilon)^2 \) squares with side length \( \varepsilon \) we get

\[
\mathbb{P}_\eta(\exists z \in \mathbb{Z}[i], \exists C \in \mathcal{C}^\eta(\delta) \text{ s.t. } \Lambda_{\varepsilon/2}(\varepsilon z) \cap C \neq \emptyset \text{ and } \mu^\eta_C(\Lambda_{\varepsilon}(\varepsilon z)) < x_\varphi)
\]

\[
\leq 4(k/\varepsilon)^2 \mathbb{P}_\eta(\exists B \in \mathcal{R}_{\Lambda_k}(\varepsilon/2) \text{ with } ||\mu^\eta_B||_{TV} < x_\varphi)
\]

\[
\leq 4(k/\varepsilon)^2 \varphi.
\]
By Theorem 5.7.2 we have that $\mu^\eta(\delta) \overset{p}{\to} \mu^0(\delta)$ in the metric $d_\ell$ for all $\delta > 0$ as $\eta \to 0$. This combined with the tightness of $|\mathcal{C}_k^0(\delta)|$, (5.41) and the Portmanteau theorem gives that

$$
P_0(\exists z \in \mathbb{Z}[i], \exists C \in \mathcal{C}_k^0(\delta) \text{ s.t. } \Lambda_{\varepsilon/2}(\varepsilon z) \cap C \neq \emptyset \text{ and } \mu_C^0(\Lambda_{\varepsilon}(\varepsilon z)) < x_\varphi) \leq 4(k/\varepsilon)^2 \varphi \quad (5.42)$$

for all $\varepsilon \in (0, \delta/10)$. We take the limit $\varphi \to 0$ in (5.42) and get

$$
P_0(\exists z \in \mathbb{Z}[i], \exists C \in \mathcal{C}_k^0(\delta) \text{ s.t. } \Lambda_{\varepsilon/2}(\varepsilon z) \cap C \neq \emptyset \text{ and } \mu_C^0(\Lambda_{\varepsilon}(\varepsilon z)) = 0) = 0, \quad (5.43)$$

which shows that $\text{supp}(\mu_C^0) + \Lambda_{\varepsilon} \supseteq C$ for all $C \in \mathcal{C}_k^0(\delta)$ with probability 1 for each fixed $\varepsilon > 0$. Thus $\text{supp}(\mu_C^0) \supseteq C$ for all $C \in \mathcal{C}^0$ with probability 1, and finishes the proof of the first statement of Theorem 5.8.1.

Since the proof of (5.38) is analogous to that of Lemma 5.7.4, we only give a sketch. Let $\delta, \varepsilon > 0, C \in \mathcal{C}^0(\delta)$ and $f : C \to \mathbb{R}$ be a continuous function with compact support. Recall the definition of $\mu_{C,\varepsilon}^0$ from the lines above Lemma 5.7.4. We set

$$
\tilde{\mu}_{C,\varepsilon,\psi}^0 := \sum_{z \in \mathbb{Z}[i] : \Lambda_{\varepsilon/2}(\varepsilon z) \cap C \neq \emptyset} \tilde{\mu}_{A(\varepsilon z, \varepsilon/2, \delta/2 - \varepsilon),\psi}^0.
$$

Note that when we replace $\mu_{A(\varepsilon z, \varepsilon/2, \delta/2 - \varepsilon)}^0$ by $\tilde{\mu}_{A(\varepsilon z, \varepsilon/2, \delta/2 - \varepsilon),\psi}^0$ in the definition of $\mu_{C,\varepsilon}^0$, we arrive to the measure $\tilde{\mu}_{C,\varepsilon,\psi}^0$. Thus for any fixed $\varepsilon > 0$ Lemma 5.8.3 shows that $\tilde{\mu}_{C,\varepsilon,\psi}^0(f)$ and $\mu_{C,\varepsilon}^0(f)$ are close to each other in $L^2$ when $\psi$ is small. In particular, $\tilde{\mu}_{C,\varepsilon,\psi}^0 \to \mu_{C,\varepsilon}^0$ weakly in probability as $\psi \to 0$.

Arguments similar to those in the proof of Lemma 5.7.4 give that $\tilde{\mu}_{C,\psi}^0$ and $\mu_{C,\varepsilon,\psi}^0$ are close to each other in total variation distance (hence in Prokhorov distance as well) with high probability when $\psi$ and $\varepsilon$ are both small.

By the proof of Theorem 5.7.2, $\mu_{C,\varepsilon}^0$ is close to $\mu_C^0$ in Prokhorov distance when $\varepsilon$ is small with high probability. Thus

$$
\tilde{\mu}_{C,\psi}^0 \approx \tilde{\mu}_{C,\varepsilon,\psi}^0 \xrightarrow{\psi \to 0} \mu_{C,\varepsilon}^0 \xrightarrow{\varepsilon \to 0} \mu_C^0,
$$

where the limits are in Prokhorov metric in probability, and $\tilde{\mu}_{C,\psi}^0 \approx \tilde{\mu}_{C,\varepsilon,\psi}^0$ means that the Prokhorov distance of these measures is small with high probability when $\varepsilon$ and $\psi$ are both small. Thus (5.38) follows, and Theorem 5.8.1 is proved. \qed

### 5.8.2 Conformal invariance and covariance

In this section we prove Theorem 5.1.4 and the stronger conformal covariance of Bernoulli percolation clusters as stated in Theorem 5.2.2.

Let us first restrict ourselves to critical site percolation on the triangular lattice. At the end of this section we will show how to obtain the weaker invariance of Theorem 5.1.4 from our general assumptions.

Recall Definition 5.3.4 of the restriction of a configuration to a bounded domain $D$. 
Theorem 5.8.5. Let, for $\eta \geq 0$, $\mathbb{P}_\eta$ denote the measure for critical site percolation on the triangular lattice. Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be a conformal map. The laws of $(f(\omega_{0,D}), f(L_{0,D}))$ and $(\omega_{f(D)}, L_{0,f(D)})$ coincide.

The conformal invariance of the continuum loop process was proved in [22, Theorem 3, item 4]. The conformal invariance of the quad crossings, follows immediately because of the measurability with respect to the loop process.

The construction of the continuum clusters and their measures was obtained in Sections 5.4 - 5.7 by approximating the cluster by boxes $\Lambda_{\varepsilon/2}(z)$. In order to prove conformal invariance / covariance we would like to approximate the clusters by conformally transformed boxes $f(\Lambda_{\varepsilon/2}(z))$. More precisely, let $\phi > 0$ and $f : \Lambda_{1+\phi} \to \hat{\mathbb{C}}$ be a conformal map. We set $D = f(\Lambda_1)$ and $D' := f(\Lambda_{1+\phi})$. Let $d_f$ denote the push-forward of the $L^\infty$ metric on $\Lambda_{1+\phi}$. That is,

$$d_f(x,y) := ||f^{-1}(x) - f^{-1}(y)||_\infty$$

for $x, y \in D'$. Note that $f$ is defined in an open neighbourhood of $\Lambda_1$ because when we approximate the cluster measures using one arm measures, we need to consider annuli whose inner square is contained in $\Lambda_1$ but which are not completely contained in $\Lambda_1$.

Clearly, $(\Lambda_{1+\phi}, d_\infty)$ and $(D', d_f)$ are isomorphic as metric spaces. Thus all the geometric constructions in Section 5.4 can be repeated for the clusters in $D$ just by applying the map $f$. We denote these analogues of the objects by an additional ‘$f$’ subscript. Thus all the statements apart from those in Section 5.4.1 remain valid if we keep the constants such as $\varepsilon, \delta$ unchanged, but add an additional subscript $f$ in the objects appearing in the claims. Moreover, the bounds in Section 5.4.1 remain valid asymptotically, as $\eta \to 0$, if we use the transformed boxes $f(\Lambda_{\varepsilon/2}(z))$ to define the relevant events because of the conformal invariance of the scaling limit.

Next note that there is a positive constant $K = K(f)$ such that $|f'(u)| \in [1/K, K]$ for $u \in \Lambda_{1+\phi/2}$. Thus $d_f$ and the $L^\infty$-metric are equivalent on $D$. As above, we add a subscript ‘$f$’ for the metrics built from $d_f$. Thus $d_{H,f}$ and $d_{P,f}$ are equivalent to $d_H$ and $d_P$ respectively, where $d_{H,f}$ and $d_{P,f}$ are built on $d_f$.

We can obtain the clusters in $D$ in two ways: via the square boxes $\Lambda_{\varepsilon/2}(z)$, that is, using the metric $L^\infty$ in $D$, or via the transformed boxes $f(\Lambda_{\varepsilon/2}(z))$, that is, using the metric $d_f$. The equivalence of the metrics implies that these two approximations provide the same continuum clusters in the scaling limit.

Now notice that the scaling limit in $D$ in terms of quad crossings is distributed like the image under $f$ of the scaling limit in $\Lambda_1$, because of the conformal invariance of quad crossing configurations. This implies that the construction in $D$, using the transformed boxes $f(\Lambda_{\varepsilon/2}(z))$, gives clusters that have the same distribution as the images of the continuum clusters in $\Lambda_1$. This proves the following theorem.

Theorem 5.8.6. For $\eta \geq 0$, let $\mathbb{P}_\eta$ denote the measure for critical site percolation on the triangular lattice. Let $\phi > 0$, $f : \Lambda_{1+\phi} \to \hat{\mathbb{C}}$ be a conformal map, and $D := f(\Lambda_1)$.

Then the laws of $\mathcal{B}^0_D$ and $f(\mathcal{B}^0_{\Lambda_1})$ are identical, where

$$f(\mathcal{B}^0_{\Lambda_1}) := \{f(B) : B \in \mathcal{B}^0_{\Lambda_1}\}.$$
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In addition to the convergence of arm measures, Garban, Pete and Schramm also proved in [38] the conformal covariance of these measures. They prove the following theorem, which is Theorem 6.7 in their paper.

**Theorem 5.8.7.** For \( \eta \geq 0 \), let \( \mathbb{P}_\eta \) denote the measure for critical site percolation on the triangular lattice. Let \( D \subseteq \mathbb{C} \) be a domain and \( f : D \rightarrow \mathbb{C} \) be a conformal map. Let \( A \subset \mathbb{C} \) be a proper annulus with piece-wise smooth boundary with \( \overline{A} \subset D \). For a Borel set \( B \subseteq f(D) \), let

\[
\mu_{1,A}^0(B) := \int_{f^{-1}(B)} |f'(z)|^{2-\alpha_1} \, d\mu_{1,A}^0(z).
\]

Then the law of \( \mu_{1,f(A)}^0 = \mu_{1,f(A)}^0(\omega_{0, f(D)}) \) and \( \mu_{1,A}^0 = \mu_{1,A}^0(\omega_{0,D}) \) coincide.

The arguments in the proof of Lemma 5.7.4 imply that approximating the cluster measures by one-arm measures of annuli of the form \( f(\Lambda_{\delta/2} \setminus \Lambda_{\varepsilon/2}) \) provides the same limit as approximating the cluster measures, in \( D \), by one-arm measures of annuli of the form \( \Lambda_{\delta/2} \setminus \Lambda_{\varepsilon/2} \). This observation and Theorem 5.8.7 imply the following result, where \( \mathcal{M}_D^0 \) denotes the collection of measures of all clusters in \( \mathcal{B}_D^0 \).

**Theorem 5.8.8.** For \( \eta \geq 0 \), let \( \mathbb{P}_\eta \) denote the measure for critical site percolation on the triangular lattice. Let \( \phi > 0 \), \( f : \Lambda_{1+\phi} \rightarrow \mathbb{C} \) be a conformal map, and \( D := f(\Lambda_1) \). Then the laws of \( \mathcal{M}_D^0 \) and \( f(\mathcal{M}_{\Lambda_1}^0) \) are identical, where, with the notation of Theorem 5.8.7,

\[
f(\mathcal{M}_{\Lambda_1}^0) := \{ \mu^0 : \mu^0 \in \mathcal{M}_{\Lambda_1}^0 \}.
\]

We are now ready to give the proofs of two of our main results, Theorems 5.2.2 and 5.1.4.

**Proof of Theorem 5.2.2.** This is a combination of Theorems 5.8.6 and 5.8.8. \( \square \)

**Proof of Theorem 5.1.4.** Note that it is sufficient to prove that the pairs

\[
(f(\mathcal{E}^0), f(\mathcal{M}^0)) \text{ and } (\mathcal{E}^0, \mathcal{M}^0)
\]

have the same distribution. This follows from a straightforward modification of the arguments above. Namely, the rotation and translation invariance and scaling covariance of the 1-arm measures under the general Assumptions I - IV follows easily from the proof of Theorem 5.3.19. See also [38, Equation (6.1) and Proposition 6.4]. \( \square \)

### 5.9 Proof of the convergence of the largest Bernoulli percolation clusters

Now we turn to the precise version and to the proof of Theorem 5.2.1.

**Theorem 5.9.1.** Let \( \mathbb{P} \) be a coupling where \( (\omega_\eta, L_\eta) \rightarrow (\omega_0, L_0) \) a.s. as \( \eta \rightarrow 0 \). Then for all \( i \in \mathbb{N} \) the \( i \)-th largest cluster \( \mathcal{M}^\eta_{(i)} \) converges in \( \mathbb{P} \)-probability to \( \mathcal{M}_{(i)}^0 \) as \( \eta \rightarrow 0 \), where \( \mathcal{M}_{(i)}^0 \) is a measurable function of \( (\omega_0, L_0) \). In particular, \( (\omega_\eta, L_\eta, \mathcal{M}^\eta_{(i)}) \rightarrow (\omega_0, L_0, \mathcal{M}_{(i)}^0) \) in distribution. The same convergence holds for the measures \( \mu_{\mathcal{M}^\eta_{(i)}} \).


Let us start with some preliminary results. Recall the definition of collections of (portions of) clusters \( \mathcal{B}_\lambda^n(\delta) \) in Section 5.6.

**Proposition 5.9.2.** Let \( \delta \in (0,1) \). For all \( \varphi > 0 \) there exist \( \eta_0, \alpha > 0 \) such that for all \( \eta < \eta_0 \):

\[
\mathbb{P}_\eta(\exists \mathcal{B}, \mathcal{B}' \in \mathcal{B}_\lambda^n(\delta) : \mathcal{B} \neq \mathcal{B}' : |\mu_\mathcal{B}^n(\Lambda_1) - \mu_\mathcal{B}'^n(\Lambda_1)| < \alpha) < \varphi.
\]

**Proof of Proposition 5.9.2.** In Chapter 3 a proof for Proposition 5.9.2 was given for bond percolation on the square lattice, however the proof also works for other models, like site percolation on the triangular lattice as noted in remark (i) after Theorem 3.1.1.

The following lemma is a complement of Lemma 5.3.14.

**Lemma 5.9.3** (Lemma 4.4 of [17]). There are positive constants \( c, C \) such that for all \( x, y > 0 \)

\[
\mathbb{P}_\eta(\exists \mathcal{B} \in \mathcal{B}_\lambda^n : |\mu_\mathcal{B}^n(\Lambda_1)| > x \text{ and } \text{diam}(\mathcal{B}) < y) < Cy^{-1} \exp(-cx/\sqrt{y})
\]

for all \( \eta < \eta_0 = \eta_0(x, y) \).

The next proposition follows easily from a combination of Lemma 5.9.3 and [17, Theorem 3.1, 3.3 and 3.6]. See also Corollary 3.2.4.

**Proposition 5.9.4.** Let \( i \in \mathbb{N} \). For all \( \varphi > 0 \) there exist \( \delta > 0, \eta_0 > 0 \) such that for all \( \eta < \eta_0 \):

\[
\mathbb{P}_\eta(\exists j \leq i : \mathcal{M}^n_{(j)} \notin \mathcal{B}^n_{\Lambda_1}(\delta)) < \varphi.
\]

**Proof of Theorem 5.9.1.** Let \( i \in \mathbb{N} \) be fixed and \( \mathbb{P} \) be a coupling such that \( (\omega_\eta, L_\eta) \to (\omega_0, L_0) \) a.s. as \( \eta \to 0 \). First we show that the \( i \)-largest clusters in the scaling limit can almost surely be defined as a function of the pair \( (\omega_0, L_0) \). Then we show that the \( i \)-th largest cluster \( \mathcal{M}^n_{(i)} \) in the discrete configuration \( \omega_\eta \) converges to the \( i \)-th largest continuum cluster.

Let \( m \in \mathbb{N} \). Theorems 5.6.1 and 5.7.2 show that the sequence of clusters \( \mathcal{B}^0_{\Lambda_1}(3^{-m}) \) and their corresponding measures \( \mu^0(3^{-m}) \) are a.s. well defined.

We define the **volume** of a continuum cluster \( \mathcal{B} \in \mathcal{B}^0_{\Lambda_1} \) as \( \mu_{\mathcal{B}}^0(\Lambda_1) \). Lemma 5.3.14 shows that the volumes of the clusters \( \mathcal{B} \in \mathcal{B}^0_{\Lambda_1}(3^{-m}) \) are a.s. finite. Moreover, Lemma 5.5.3, together with the tightness of the number of excursions in \( \Lambda_1 \), of diameter at least \( 3^{-m} \), gives that \( h^0(3^{-m}) := |\mathcal{B}^0_{\Lambda_1}(3^{-m})| \) is a.s. finite. Thus we can reorder the sequence of clusters \( \mathcal{B}^0(3^{-m}) \) in decreasing order by their volume. We break ties in some deterministic way. However, we will see below that ties occur with probability 0. Let \( \mathcal{M}^n_{(j)}(3^{-m}) \) denote the \( j \)-th cluster in this new ordering.

Let \( \varphi > 0 \) arbitrary. Take \( \alpha \) and \( \eta_0 \) as in Proposition 5.9.2. Then, for \( \eta < \eta_0 \)

\[
\mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathcal{B}^0_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}' : |\mu_{\mathcal{B}}^0(\Lambda_1) - \mu_{\mathcal{B}'}^0(\Lambda_1)| < \alpha/2)
\]

\[
\leq \mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathcal{B}^0_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}' : |\mu_{\mathcal{B}}^0(\Lambda_1) - \mu_{\mathcal{B}'}^0(\Lambda_1)| < \alpha)
\]

\[
+ \mathbb{P}(\exists j \leq h^0(3^{-m}) : |\mu_{\mathcal{B}^n_{(j)}}^n(\Lambda_1) - \mu_{\mathcal{B}^n_{(j)}}^n(\Lambda_1)| > \alpha/4)
\]

\[
\leq \varphi + \mathbb{P}(\exists j \leq h^0(3^{-m}) : |\mu_{\mathcal{B}^n_{(j)}}^n(\Lambda_1) - \mu_{\mathcal{B}^n_{(j)}}^n(\Lambda_1)| > \alpha/4).
\]
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The second term in the right hand side of (5.44) tends to 0 as \( \eta \to 0 \), since \( h^0 \) is a.s. finite and \( \mu^0(3^{-m}) \to \mu^0(3^{-m}) \) in probability by Theorem 5.7.2. Since \( \varphi > 0 \) was arbitrary, this shows that

\[
\mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathcal{B}^0_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}', |\mu^0_{\mathcal{B}}(\Lambda_1) - \mu^0_{\mathcal{B}'}(\Lambda_1)| = 0) = 0,
\]

that is, there are no ties in the ordering above with probability 1.

Now we show that, for all \( j \leq i \),

\[
\mathbb{P}(\exists m_0 \in \mathbb{N} \text{ s.t. } \mathcal{M}^0_{(j)}(3^{-m_0}) = \mathcal{M}^0_{(j)}(3^{-m}) \text{ for all } m \geq m_0) = 1. \quad (5.45)
\]

Suppose the contrary, and let \( j_0 \) be the smallest \( j \leq i \) so that (5.45) fails. Let

\[
E = \{ \exists m_0 \in \mathbb{N} \text{ s.t. } \mathcal{M}^0_{(j_0)}(3^{-m_0}) = \mathcal{M}^0_{(j_0)}(3^{-m}) \text{ for all } m \geq m_0 \},
\]

and \( \varphi = \mathbb{P}(E) > 0 \).

The definition of \( j_0 \) implies that, on the event \( E \), there is a sequence of clusters \( (\tilde{\mathcal{B}}_n)_{n \geq 1} \subseteq \mathcal{B}^0_{\Lambda_1} \) so that \( \text{diam}(\tilde{\mathcal{B}}_n) \to 0 \) as \( n \to 0 \), and \( \mu_{\tilde{\mathcal{B}}_n}(\Lambda_1) \) is increasing. Take \( \delta > 0 \) so small that

\[
\mathbb{P}(\exists n \in \mathbb{N} \text{ s.t. } \tilde{\mathcal{B}}_n \in \mathcal{B}^0_{\Lambda_1}(\delta), E) > \varphi/2.
\]

Since \( \mu_{\tilde{\mathcal{B}}_n}(\Lambda_1) \) is increasing, the equation above combined with Lemma 5.8.2 shows that there is \( x > 0 \) so that

\[
\mathbb{P}(\lim_{n \to \infty} \mu_{\tilde{\mathcal{B}}_n}(\Lambda_1) > x, E) > \varphi/4.
\]

Since \( \text{diam}(\tilde{\mathcal{B}}_n) \to 0 \) as \( n \to 0 \), the above implies that there are deterministic sequences \( \delta_n, \delta'_n \) tending to 0 as \( n \to 0 \) so that

\[
\mathbb{P}(\exists \mathcal{B} \in \mathcal{B}^0_{\Lambda_1} \text{ with } \mu_{\mathcal{B}}(\Lambda_1) > x/2 \text{ and } \text{diam}(\mathcal{B}) \in (\delta_n, \delta'_n)) \geq \varphi/8.
\]

Theorem 5.5.1 and 5.7.2 implies that for all \( n \geq 0 \) there is \( \eta_0(n) \) so that

\[
\mathbb{P}(\exists \mathcal{B} \in \mathcal{B}^0_{\Lambda_1} \text{ with } \mu_{\mathcal{B}}(\Lambda_1) > x/4 \text{ and } \text{diam}(\mathcal{B}) \in (\delta_n/2, 2\delta'_n)) \geq \varphi/16,
\]

for all \( \eta \leq \eta_0(n) \). Since \( \delta_n \to 0 \) as \( n \to 0 \), taking \( n \) large enough, we get a contradiction with Lemma 5.3.14. Hence (5.45) is proved, and for each \( j \leq i \) we set \( \mathcal{M}^0_{(j)} := \mathcal{M}^0_{(j)}(3^{-m_0}) \) where \( m_0 \) as in the event on the left hand side of (5.45).

It remains to show that \( \mathcal{M}^0_{(j)} \) converges in probability to \( \mathcal{M}^0_{(i)} \) as well as their measures. Let \( \varepsilon, \alpha > 0 \) and \( m > 0 \), first we check that

\[
\mathbb{P}(\text{d}_H(\mathcal{M}^\eta_{(i)}(\mathcal{M}^0_{(i)}), > \varepsilon)
\]

\[
\leq \mathbb{P}(\exists j \leq i : \mathcal{M}^\eta_{(j)} \neq \mathcal{M}^0_{(j)}(3^{-m}))
\]

\[
+ \mathbb{P}(\exists j \leq i : \mathcal{M}^\eta_{(j)} \neq \mathcal{M}^0_{(j)}(3^{-m}))
\]

\[
+ \mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathcal{B}^0_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}', |\mu_{\mathcal{B}}(\Lambda_1) - \mu_{\mathcal{B}'}(\Lambda_1)| < \alpha)
\]

\[
+ \mathbb{P}(\exists \mathcal{B}, \mathcal{B}' \in \mathcal{B}^0_{\Lambda_1}(3^{-m}) : \mathcal{B} \neq \mathcal{B}', |\mu_{\mathcal{B}}(\Lambda_1) - \mu_{\mathcal{B}'}(\Lambda_1)| < \alpha)
\]

\[
+ \mathbb{P}(\exists k \leq h^0(3^{-m}) : |\mu_{\mathcal{B}^\eta_k}(\Lambda_1) - \mu_{\mathcal{B}^\eta_k}(\Lambda_1)| > \alpha/3)
\]

\[
+ \mathbb{P}(\exists k \leq h^0(3^{-m}) : d_H(\mathcal{B}^\eta_k, \mathcal{B}^0_k) > \varepsilon),
\]

\[
(5.46)
\]
where \( B^n_k \) and \( B^0_k \) are the \( k \)-th cluster in the ordering used in the proofs of Theorem 5.5.1 and 5.7.2 of the clusters in \( B^n_{\Lambda_1}(3^{-m}) \) and in \( B^0_{\Lambda_1}(3^{-m}) \), respectively.

We justify (5.46) as follows. On the complement of the first two events on the right hand side of (5.46), all of the \( i \) largest clusters at scale \( \eta \) and \( 0 \) (i.e. in the scaling limit) have diameter at least \( 3^{-m} \). On the complement of the third and fourth event on the right hand side of (5.46), the normalized volumes of the different clusters with diameter at least \( 3^{-m} \) are at least \( \alpha \) apart at both scales \( \eta \) and \( 0 \). Thus on the complement of the first five events on the right hand side of (5.46) the ordering according to their volume of the \( k \) largest clusters at scale \( \eta \) and \( 0 \) coincide, that is for all \( j \leq i \), there is a unique \( k_j \leq h^0(3^{-m}) \) so that \( \mathcal{M}^n_{(j)} = B^n_{k_j} \) and \( \mathcal{M}^0_{(j)} = B^0_{k_j} \). This together with the last term in the right hand side of (5.46) proves (5.46).

Let \( \varphi > 0 \) arbitrary. By (5.45) and Proposition 5.9.4, we find \( m \) and \( \eta_0 > 0 \) such that the first and second term on the right hand side of (5.46) are less than \( \varphi/6 \) for all \( \eta < \eta_0 \). Then we use the bounds in (5.44) and Proposition 5.9.2 and find \( \alpha, \eta_1 > 0 \) so that the third and fourth term on the right hand side of (5.46) are less than \( \varphi/6 \) for all \( \eta < \eta_1 \). Finally, we apply Theorem 5.5.1 for the fifth term and Theorem 5.7.2 for the sixth term and deduce that \( \limsup_{\eta \to 0} \mathbb{P}(d_H(\mathcal{M}^n_{(i)}, \mathcal{M}^0_{(i)}) > \varepsilon) < \varphi \). Since \( \varphi \) and \( \varepsilon \) were arbitrary, this shows that \( \mathcal{M}^n_{(i)} \to \mathcal{M}^0_{(i)} \) in probability as \( \eta \to 0 \).

The proof for the convergence of normalized counting measures goes in a similar way: notice that if we replace the fifth term on the right hand side of (5.46) with

\[
\mathbb{P}(\exists j \leq h^0(3^{-m}) : d_P(\mu^n_{B^n_{k_j}}, \mu^0_{B^0_{k_j}}) > \alpha/3),
\]

then we get an upper bound for the probability \( \mathbb{P}(\exists j \leq i : d_P(\mu^n_{\mathcal{M}^n_{(j)}}, \mu^0_{\mathcal{M}^0_{(j)}}) > \alpha/3) \). This completes the proof of Theorem 5.9.1. \( \Box \)