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A CRITICAL ACCOUNT OF PERTURBATION ANALYSIS OF MARKOV CHAINS

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Abstract. Perturbation analysis of Markov chains provides bounds on the effect that a change in a Markov transition matrix has on the corresponding stationary distribution. This paper compares and analyzes bounds found in the literature and introduces a new bound. We provide for the first time an analysis of the relative error of these bounds. Specifically, we show that condition number bounds have a non-vanishing relative error as the size of the perturbation tends to zero. Our new perturbation bound will have the desirable feature that the relative error vanishes as the size of the perturbation tends to zero. We discuss a series of examples to illustrate the applicability of the various bounds. Specifically, we address the question on how the bounds behave as the size of the system grows.

Key words. Markov chains, perturbation bounds, condition number, strong stability, series expansion, queuing

AMS subject classifications. Primary: 60J10; Secondary: 15A12; 15A18

1. Introduction. Perturbation analysis of Markov chains studies the effect a perturbation of a Markov transition matrix has on the stationary distribution of the chain. This line of research dates back to Schweitzer's pioneering paper [32]. Consider a Markov chain with discrete state space S , transition probability matrix P , and unique stationary distribution π_P . Furthermore, let Δ be a matrix such that $R = P + \Delta$ (i.e., $\Delta = R - P$) is a Markov chain with unique stationary distribution π_R . Provided that R has unique stationary distribution π_R , perturbation analysis of Markov chains (PAMC) addresses the following question: what is the effect of perturbing P by Δ on the stationary distribution of the chain? More specifically, PAMC theory studies bounds of the type

$$(1.1) \quad \|\pi_R^\top - \pi_P^\top\|_a \leq \kappa \|R - P\|_b,$$

where $\|\cdot\|_a$ and $\|\cdot\|_b$ are appropriate norms, for details see Section 2.2 where the choice of norms is discussed in detail, π_R^\top denotes the transpose of column vector π_R and κ is the so-called *condition number*. PAMC is a field of active research [4, 25, 30, 31, 33, 36] and various condition numbers have been proposed in the literature [11]. To simplify the notation, we will in the following suppress the explicit notation of the subscripts indicating the specific type of norm whenever this causes no confusion.

The above *condition number bound* (CNB) has the attractive feature that it provides a uniform bound on the ball $\{R \in \mathbb{P}(S) : \|R - P\| < \epsilon\}$, where $\mathbb{P}(S)$ denotes the

set of all Markov transition matrices R on S such that R possesses a unique stationary distribution. More specifically, it holds for condition number κ that

$$(1.2) \quad \sup_{\{R \in \mathbb{P}(S) : \|R - P\| < \epsilon\}} \|\pi_R^\top - \pi_P^\top\| \leq \epsilon \kappa.$$

It is worth noting that in (1.2) existence of π_R is assumed. As will show in this paper, sufficient conditions can be established such that existence of π_P together with $\|R - P\|$ being sufficiently small already implies existence of π_R .

Inspired by (1.2), we define the *robust sensitivity* of π_P by

$$(1.3) \quad \frac{d\|\pi_P\|}{d\|P\|} := \lim_{\epsilon \downarrow 0} \sup_{\{R \in \mathbb{P}(S) : \|R - P\| < \epsilon\}} \frac{\|\pi_R^\top - \pi_P^\top\|}{\|R - P\|},$$

provided the limit exists. The robust sensitivity can be interpreted as a non-discriminatory sensitivity as it assumes no model for possible perturbations. Dividing inequality (1.2) by ϵ and letting $\epsilon \downarrow 0$ shows that condition number κ can be interpreted as an upper bound for the robust sensitivity of π_P .

Next to the robust sensitivity we will also consider perturbation analysis of convex combinations of Markov chains. More specifically, let R, P be two Markov kernels defined on the same state space. Then the convex combination of both kernels

$$(1.4) \quad P(\theta) = (1 - \theta)P + \theta R, \quad \theta \in [0, 1],$$

is a well-defined Markov kernel. Note that $P(0) = P$ and $P(1) = R$. In perturbation analysis of $P(\theta)$ we are interested in the effect of changing θ from 0 to some value $0 < \theta \leq 1$, where $P(0) = P$. By linearity of norms,

$$(1.5) \quad \|P(\theta) - P\| = \theta \|R - P\|,$$

for $\theta \in [0, 1]$. Provided that

$$\|\pi_{P(\theta)}^\top - \pi_P^\top\| \leq \|P(\theta) - P\| \kappa,$$

we arrive by (1.5) at

$$\|\pi_{P(\theta)}^\top - \pi_P^\top\| \leq \theta \|R - P\| \kappa,$$

which provides an uniform bound for $\pi_{P(\theta)}$ as a function of θ .

Such a functional approximation is of interest if $P(\theta)$, for $\theta \in [0, 1]$, has a clear interpretation. We will illustrate this by a queueing model with breakdowns, where θ models the probability of a breakdown. An interesting observation is that in the parametrized model we establish conditions for stability of a mixture of a stable (no breakdowns) and an unstable (only breakdowns) Markov chain. More specifically, we provide a lower bound of the stability of $P(\theta)$. Moreover, for the parametric model (1.4) it follows from (1.5) that the perturbation bounds via conditioning numbers are linear in θ . As the effect of changing θ in (1.4) on the stationary distribution of $P(\theta)$ is typically non-linear, this hints at the fact that this type of perturbation bounds will only be meaningful for small perturbations.

In this paper we review the existing perturbation bounds via conditioning numbers. In addition we will also review (and improve) bounds obtained by the strong stability approach and the series expansion technique. Specifically, we will compare

the quality of the bounds. For example, if the bound is expressed in the maximum absolute value norm, then $\|\pi_R^\top - \pi_P^\top\| \leq 2$ and any bound yielding a value larger than this trivial bound is meaningless. As we will show by means of simple examples, condition number bounds are typically ill-behaved. In addition, we study the limit of the bound in terms of the size of the network. Bounds may grow linearly or even quadratically in the size of the network, implying that they are only informative for small networks. To overcome this curse of network size, we provide bounds that do not suffer from the dimension of the problem. Moreover, we will show that the relative error of existing bounds in predicting $\|\pi_R^\top - \pi_P^\top\|$ does converge to some finite non-zero value as the size of the perturbation $\|R - P\|$ tends to zero. In this paper we will provide an alternative bound based on a series expansion that overcomes this drawback and has a provable rate of convergence of the relative error to zero. Having provided a detailed comparison between perturbation bounds, we will analyze the behavior of the perturbation bounds for a realistic queueing example.

The paper is organized as follows. In Section 2 the perturbation bounds are presented. Examples are discussed in Section 3. Section 4 is devoted to perturbation bounds for the M/G/1 queue with breakdowns. Other than the small numerical examples reported in the literature, the queueing system will be analyzed for the case of a large but finite state-space and for the infinite dimensional case.

2. Perturbation Analysis. Throughout this paper we will consider Markov chains defined on an at most denumerable state space $S = \{0, 1, \dots, n-1\}$, $n \geq 2$, and consisting of one closed communicating class of states with possible transient states. Unless indicated otherwise we follow the convention that vectors are column vectors.

2.1. Preliminaries and Basic Definitions. If $P = (P_{ij})_{i,j \in S}$ is a Markov transition matrix of some Markov chain $\{X_k\}$, then $P_{ij} = \mathbb{E}[X_{k+1} = j | X_k = i]$ for $i, j \in S$ and $k \in \mathbb{N}$. Sometimes $P(i, j) := P_{ij}$ is used instead for notation clarity. Further, let $f \in \mathbb{R}^S$ be the reward vector where f_i is the reward for being in state $i \in S$. With these definitions, one obtains

$$(2.1) \quad \mu^\top P f = \sum_{i,j \in S} \mu_i P_{ij} f_j = \sum_{i \in S} \mathbb{E}[f_{X_1} | X_0 = i] \mu_i$$

as the expected reward after one transition provided the Markov chain is started with initial distribution μ . For more details we refer to [22, 23].

In the following denote the stationary distribution of P by π_P , and we let D_P denote the deviation matrix of P , which is given by

$$(2.2) \quad D_P = \sum_{k=0}^{\infty} (P^k - \Pi_P) = (I - P + \Pi_P)^{-1} - \Pi_P,$$

with Π_P being the ergodic projector of P , i.e., the matrix with rows identical to π_P^\top . Note that provided that the sum over $(P^k - \Pi_P)$ converges in (2.2), it follows from simple algebra that $I - P + \Pi_P$ is invertible. The deviation matrix is also referred to as the group inverse. Moreover, the deviation matrix is an instance of the generalized inverse of $I - \Pi_P$; see [29] for an early reference. As Hunter demonstrates in [19], the generalized inverse plays a major role in perturbation analysis. While it can be defined in various ways, we will work here with the deviation matrix and its representation as this allows for a convenient mathematical analysis.

In our analysis we will frequently work with the *taboo kernel* of a Markov transition matrix P . In [21] a very elegant and flexible way for obtaining a taboo kernel is described. For this let h be a non-negative vector and σ a probability measure on S . The taboo kernel of P with respect to h and σ is defined as

$$(2.3) \quad T := P - h\sigma^\top$$

with $h\sigma^\top$ denoting the matrix product of vector h and σ , i.e., $h\sigma^\top$ is a square matrix. For example, let

$$h = (P(0,0), P(1,0), P(2,0), \dots)^\top$$

denote the first column of P , and let $\sigma = (1, 0, 0, \dots)^\top$, then

$$T = P - h\sigma^\top = \begin{cases} P(i,j) & j > 0 \\ 0 & \text{otherwise} \end{cases}.$$

In words, T is a degenerate transition kernel that avoids entering state zero which is obtained by setting the first column of P to zero. Alternatively, letting $\sigma = (1, 0, 0, \dots)^\top$ and

$$h = (P(0,1), P(0,1), P(0,2), \dots)^\top,$$

then $T = P - \sigma h^\top$ is a degenerate transition kernel that never leaves state zero, which is obtained by setting the first row of P to zero. In the following we write ${}_iP$ for the degenerate transition kernel that avoids entering state i which is obtained by setting the i th column of P to zero, i.e., letting $\sigma = (0, \dots, 0, 1, 0, \dots)^\top$, where the entry 1 is at the i th position, and h the i th column of P . The taboo kernel ${}_iP$ provides a sufficient condition for positive recurrence of P . The precise statement is provided in the following corollary.

COROLLARY 1. *Let P be irreducible. If for at least one $i \in S$ it holds that $\|{}_iP\| < 1$, then P is positive recurrent.*

Proof. The first recurrence time to state i is given by element (i, i) of $\sum ({}_iP)^n$, which is finite due to the norm condition. Therefore, state i is positive recurrent. From irreducibility of P it follows that all states are positive recurrent. \square

Elaborating on the taboo kernel of P , the deviation matrix can alternatively be written as

$$(2.4) \quad D_P = (I - \Pi_P) \sum_{n=0}^{\infty} T^n (I - \Pi_P),$$

see [18, 20], where we note that

$$(2.5) \quad (I - T)^{-1} = \sum_{n=0}^{\infty} T^n,$$

under the assumption that $\|T\| < 1$ for some matrix norm $\|\cdot\|$. Appropriate norms are discussed in the following.

For $x \in \mathbb{R}^S$, we denote by $\|x\|_\infty$ the maximum absolute value (also referred to as infinity norm) and by $\|x\|_1$ the sum of absolute value (a.k.a. L_1 norm). For a matrix $A \in \mathbb{R}^{S \times S}$, we let $\|A\|_\infty$ denote the maximum absolute row sum. In the paper we

will frequently work with the weighted supremum norm, also called v -norm, denoted by $\|\cdot\|_v$, where the v -norm of a matrix $A \in \mathbb{R}^{S \times S}$ is given by

$$(2.6) \quad \|A\|_v = \sup_i \frac{1}{v(i)} \sum_{j \in S} |A(i, j)| v(j),$$

for v such that $v(0) = 1$ and $v(i) \geq 1$ for all $i \in S$. In the following we let

$$v(i) = \alpha^i, \quad i \in S,$$

with α some unspecified constant $\alpha \in [1, \infty)$. By convention, vectors $x \in \mathbb{R}^S$ are column vectors so that the above definition implies that for $x \in \mathbb{R}^S$ we have

$$(2.7) \quad \|x^\top\|_v = \sum_{k \in S} v(k) |x_k|,$$

and

$$(2.8) \quad \|x\|_v = \sup_{i \in S} \frac{|x_i|}{v(i)}.$$

The distinction is motivated by the application to Markov chains, where probability measures are row vectors to which norm (2.7) applies and reward functions are column vectors to which norm (2.8) applies. Specifically, applying the v -norm to (2.1) one readily obtains

$$\|\mu^\top P f\|_v \leq \|\mu^\top\|_v \|P\|_v \|f\|_v.$$

Note that for a (possibly signed) measure μ on \mathbb{R}^S the v -norm of μ^\top , for $v \equiv 1$, coincides with total variational norm. Note further that $v \geq 1$ implies for $x \in \mathbb{R}^S$ that $\|x\|_v \leq \|x\|_\infty$. In addition, for a measure μ we have that $\|\mu^\top\|_1 \leq \|\mu^\top\|_v$.

To illustrate the efficiency of the bounds we will use throughout the paper three different types of Markov chains introduced in the following.

EXAMPLE 1.

Small Network: Let $S = \{0, 1\}$ and

$$P^s = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix},$$

with $p, q \in (0, 1)$. Since $|1-p-q| < 1$ we get the stationary distribution:

$$\pi_{P^s} = \frac{1}{p+q} (q, p)^\top.$$

The deviation matrix is given by:

$$D_{P^s} = \frac{1}{(p+q)^2} \begin{pmatrix} p & -p \\ -q & q \end{pmatrix}.$$

Ring Network: The next example that we will discuss is that of a ring, introduced in the following. Let $S = \{0, \dots, n-1\}$ and for any $n \geq 2$,

$$P^\circ(n) = \begin{pmatrix} 1-2b & b & 0 & 0 & \dots & b \\ b & 1-2b & b & 0 & \dots & 0 \\ 0 & b & 1-2b & b & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & b & 1-2b & b \\ b & 0 & \dots & 0 & b & 1-2b \end{pmatrix},$$

with $b \in (0, 1/2]$. We get the stationary distribution:

$$\pi_i^\circ(n) = \frac{1}{n}, \quad \text{for } i \in S.$$

For the deviation matrix, we obtain:

$$D^\circ(n) := D_{P^\circ(n)} = \begin{pmatrix} d_0 & d_1 & d_2 & \dots & d_{n-1} \\ d_{n-1} & d_0 & d_1 & \dots & d_{n-2} \\ d_{n-2} & d_{n-1} & d_0 & \dots & d_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & d_3 & \dots & d_0 \end{pmatrix},$$

where

$$(2.9) \quad d_i = \frac{(n-1)(n+1)}{12bn} - \frac{(n-i)i}{2bn} \quad \text{for } i \in S.$$

Furthermore, $\sum_{i=0}^{n-1} d_i = 0$. Equivalently, $D^\circ(n)$ can be expressed as

$$D^\circ(n) = \left(\tilde{D}_{ij}(n) \right)_{i,j \in S},$$

where

$$(2.10) \quad \tilde{D}_{ij}(n) = d_{(j-i)(\text{mod } n)+1} = \frac{(n-1)(n+1)}{12bn} - \frac{\{n - (j-i)(\text{mod } n)\}\{(j-i)(\text{mod } n)\}}{2bn}.$$

Star Network: The third example considered is the Star Network with state space $S = \{0, \dots, n-1\}$. For $n \geq 2$ let

$$P^*(n) = \begin{pmatrix} 1-\beta & \frac{\beta}{n-1} & \frac{\beta}{n-1} & \frac{\beta}{n-1} & \dots & \frac{\beta}{n-1} \\ 1-\gamma & \gamma & 0 & 0 & \dots & 0 \\ 1-\gamma & 0 & \gamma & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1-\gamma & 0 & \dots & 0 & \gamma & 0 \\ 1-\gamma & 0 & \dots & 0 & 0 & \gamma \end{pmatrix},$$

for $\beta \in (0, 1]$ and $\gamma \in [0, 1)$. Following [13], the stationary distribution is given by

$$\pi_i^*(n) = \begin{cases} \frac{1-\gamma}{1-\gamma+\beta} & \text{for } i = 0, \\ \frac{\beta}{(n-1)(1-\gamma+\beta)} & \text{for } i > 0. \end{cases}$$

For the deviation matrix, we obtain:

$$D^*(n) = \left(\begin{array}{c|c} \frac{\beta}{(1-\gamma+\beta)^2} & -\frac{\beta}{(1-\gamma+\beta)^2(n-1)} \bar{\mathbf{1}}^T \\ \hline -\frac{(1-\gamma)}{(1-\gamma+\beta)^2} \bar{\mathbf{1}} & \frac{1}{(1-\gamma)} I - \frac{\beta\{(1-\gamma)+(1-\gamma+\beta)\}}{(1-\gamma)(1-\gamma+\beta)^2(n-1)} \bar{\mathbf{1}} \bar{\mathbf{1}}^T \end{array} \right),$$

where $\bar{\mathbf{1}} = [1, \dots, 1]^T$ of size $n-1$ and I denotes the $(n-1) \times (n-1)$ identity matrix.

We call T *proper* if $\|T\| < 1$. Provided that T defined in (2.3) is proper, the v -norm of π_P^\top can be bounded by

$$(2.11) \quad \|\pi_P^\top\|_v \leq \frac{\pi_P^\top h}{1 - \|T\|_v},$$

see [21].

The idea behind considering T rather than P , is that T might be constructed in such a way that the norm of T is strictly less than one. The following example illustrates the effect on $\|T\|$ from either removing the first column or first row. Note that removing the second column or second row would lead to again other values of $\|T\|$ and even the matrix indices can be respecified to obtain different values of $\|T\|$.

EXAMPLE 2. *For the Small Network, i.e., $P = P^s$, we find after removing the first column*

$$\|T\|_v = \max\{\alpha p, 1 - q\}.$$

Removing the first row leads to

$$\|T\|_v = \max\left\{0, \frac{(1 - \alpha)q}{\alpha} + 1\right\}.$$

For the Ring and the Star networks we present the resulting norms for $\|T\|_v$ (including the 1-norm) and $\|T\|_\infty$, respectively, in Table 1 and Table 2.

Removing:	Ring (i.e., $P = P^\circ(n)$)	Star (i.e., $P = P^*(n)$)
1st row of P	$\frac{b}{\alpha} + 1 - 2b + \alpha b$	$\frac{b}{\alpha} + 1 - 2b + \alpha b$
1st column of P	$\max\{\alpha b + \alpha^{n-1} b, \frac{b}{\alpha} + 1 - 2b + b\alpha\}$	$\max\left\{\gamma, \frac{\alpha\beta}{n-1} \frac{1-\alpha^{n-1}}{1-\alpha}\right\}$

TABLE 1

The v -norm for different choices for T (including the 1-norm).

Note that letting α tend to one, the values in Table 1 yield values for the 1-norm as well.

Removing:	Ring (i.e., $P = P^\circ(n)$)	Star (i.e., $P = P^*(n)$)
1st row of P	1	1
1st column of P	$\max\{2b, 1\} \geq 1$	$\max\{\gamma, \beta\}$

TABLE 2

The infinity norm for different choices for T .

In the following we discuss a general way of choosing T . Let $P_{\bullet j}$ denote the j -th column of P . For a column vector x we let $\|x\|_{\inf} = \inf_i |x_i|$. We denote the j th unit vector by e_j , i.e., e_j has all elements zero except for the j -th element which is equal to 1.

LEMMA 2.1. *Let P be a Markov transition matrix on S . Let j^* be the column index with maximal value $\|P_{\bullet j}\|_{\inf}$. If $\|P_{\bullet j^*}\|_{\inf} > 0$, let $h = P_{\bullet j^*}$ and $\sigma = e_{j^*}$, then for T defined as in (2.3) it holds that $\|T\|_v < 1$, where $v \equiv 1$.*

Proof. Without loss of generality assume that after appropriate relabeling of the states $j^* = 0$. Let $\|P_{\bullet 0}\|_v = q > 0$. Removing the first column from P thus decreased the row sum of each row of P by at least q , which implies the desired result. \square

2.2. The Choice of Norms in Perturbation Analysis. Several condition numbers have been proposed in the literature, see [11] for an overview. We keep the numbering as in [11], where seven different condition numbers were discussed. Moreover it is shown in [11] that condition numbers κ_3 and κ_6 , to be defined presently, were outperforming the other condition numbers, while the choice between κ_3 and κ_6 depends on the choice of norms. Condition number κ_3 is given by [14, 24]:

$$(2.12) \quad \kappa_3(P) = \frac{\max_j(D_P(j, j) - \min_i D_P(i, j))}{2}$$

and the resulting bound applies to the infinity norms:

$$\|\pi_R - \pi_P\|_\infty \leq \kappa_3(P)\|R - P\|_\infty.$$

Alternatively, condition number κ_6 in [33] is given by:

$$(2.13) \quad \kappa_6(P) = \frac{1}{2} \max_{i,j} \sum_{k=0}^{n-1} |D_P(i, k) - D_P(j, k)|,$$

and the resulting bound applies to the 1-norm:

$$\|\pi_R^\top - \pi_P^\top\|_1 \leq \kappa_6(P)\|R - P\|_\infty.$$

EXAMPLE 3. *The condition numbers for the Markov chains introduced in Example 1 are as follows:*

$$\kappa_3(P^s) = \frac{1}{2(p+q)} \quad \text{and} \quad \kappa_6(P^s) = \frac{1}{p+q},$$

$$\kappa_3(P^\circ(n)) = \frac{\lfloor \frac{n}{2} \rfloor (n - \lfloor \frac{n}{2} \rfloor)}{4bn},$$

$$\kappa_6(P^\circ(n)) = \frac{1}{2} \sum_{k=0}^{n-1} \left| D_{P^\circ(n)} \left(\left\lfloor \frac{n}{2} \right\rfloor + 1, k - 1 \right) - D_{P^\circ(n)}(1, k - 1) \right|,$$

and

$$\kappa_3(P^*(n)) = \frac{1}{2(1-\gamma)} \quad \text{and} \quad \kappa_6(P^*(n)) = \frac{1}{1-\gamma}.$$

It is worth noting that $\kappa_3(P^\circ(n))$ grows linearly in n . As the condition number applies to the infinity norm of $\pi_R^\top - \pi_P^\top$, which is bounded by 2, the bound becomes trivial for large n . For the Star Network, κ_3 and κ_6 do not depend on n but become trivial for γ close to 1.

The fact that κ_3 and κ_6 behave so different for the Ring and the Star networks stems from the fact that both condition numbers are defined via the deviation matrix. The elements of the deviation matrix are related to recurrence times of the corresponding Markov chain, see [29, 19]. Specifically, in the Ring Network the length of a path from, say, node 0 to node $\lfloor n/2 \rfloor$ grows with n , whereas in the Star Network any node can be reached from any other node in 2 steps.

It is known that $\kappa_3(P) < \kappa_6(P)$ (in fact it holds that $2\kappa_3(P) \leq \kappa_6(P)$), see [24]. Note that this inequality implies that $\kappa_6(P^\circ(n))$ tends for the Ring Network to infinity as well. As we will discuss in the subsequent section, $\kappa_6(P)$ may be preferable to $\kappa_3(P)$ in case bounds on perturbations of expected rewards are considered. In [24] it is shown that $\kappa_3(P) \geq (n-1)/(2n)$, with n being the size of transition matrix, and a Markov chain is provided for which equality is reached.

In bounding perturbations it is important to understand how a perturbation of the Markov chain effects the steady-state reward. Put differently, using the notation as already introduced in the introduction, relating a perturbation bound for $\|\pi_R^\top - \pi_P^\top\|_v$ to that of $|\pi_R^\top f - \pi_P^\top f|$ is of importance in applications, i.e., one seeks to find conditions under which

$$|\pi_R^\top f - \pi_P^\top f| \leq \|\pi_R^\top - \pi_P^\top\|_a \|f\|_c,$$

for $\|\cdot\|_a$ and $\|\cdot\|_c$ appropriate norms. How the introduced norms can be combined in various ways for bounding perturbations is presented in the following lemma.

LEMMA 2.2. *For arbitrary measures $\tilde{\mu}$ and μ on \mathbb{R}^S and cost function $f \in \mathbb{R}^S$ it holds that*

$$|\tilde{\mu}^\top f - \mu^\top f| \leq \begin{cases} \|\tilde{\mu}^\top - \mu^\top\|_1 \|f\|_\infty \\ \|\tilde{\mu}^\top - \mu^\top\|_\infty \|f\|_1 \\ \|\tilde{\mu}^\top - \mu^\top\|_v \|f\|_v \end{cases},$$

for all f .

Proof. By simple algebra,

$$|\tilde{\mu}^\top f - \mu^\top f| \leq \sum_i |\tilde{\mu}_i - \mu_i| |f_i| \leq \sup_j |f_j| \sum_i |\tilde{\mu}_i - \mu_i| = \|\tilde{\mu}^\top - \mu^\top\|_1 \|f\|_\infty.$$

The second inequality follows from the same line of argument. For the last inequality note that

$$\begin{aligned} |\tilde{\mu}^\top f - \mu^\top f| &\leq \sum_i |\tilde{\mu}_i - \mu_i| |f_i| \\ &= \sum_i |\tilde{\mu}_i - \mu_i| v_i \frac{|f_i|}{v_i} \\ &\leq \left(\sup_j \frac{|f_j|}{v_j} \right) \sum_i |\tilde{\mu}_i - \mu_i| v_i \\ &= \|\tilde{\mu}^\top - \mu^\top\|_v \|f\|_v, \end{aligned}$$

which concludes the proof. \square

In this paper we study the case that μ in Lemma 2.2 is a stationary distribution. Lemma 2.2 illustrates that there is a trade-off in the choice of norms. Indeed, since $\|\pi_R^\top - \pi_P^\top\|_\infty \leq \|\pi_R^\top - \pi_P^\top\|_1$ it seems attractive to ask for perturbation bounds on $\|\pi_R^\top - \pi_P^\top\|_\infty$. The downside is that this choice effects the norm of the reward vector. To illustrate this, let P be the transition kernel of an M/M/1/N queue, where N is the size of the buffer of the queue including the service place, and suppose that we are

interested in the effect replacing P by R has one the stationary queue length. More specifically, let $f_i(s) = s$, for $s \in S = \{0, 1, \dots, N\}$, and note that

$$\|f_i\|_1 = \frac{N(N+1)}{2} > N = \|f_i\|_\infty.$$

In the light of Lemma 2.2, in bounding $|\pi_P^\top f_i - \pi_R^\top f_i|$ the smaller bound on the norm distance of $\pi_R^\top - \pi_P^\top$ by applying the infinity norm is outweighed by the increase in norm for the reward. If, on the other side, one is only interested in an overflow probability, i.e., $f_p(s) = 0$ for $s < N$ and $f_p(N) = 1$, then $\|f_p\|_1 = \|f_p\|_\infty = 1$ and the infinity-norm bound for $\pi_R^\top - \pi_P^\top$ is appropriate. To underpin our argument by an example from a different application area, consider the analysis of the ‘wisdom of crowds’ phenomenon in social networks, [13]. Here, f represents a belief vector with bounded support, i.e., $f(s) \in [a, b]$ for $a < b \in \mathbb{R}$, and $\pi_P^\top f$ is the consensus reached in the social network modeled by P . Then,

$$\|f\|_\infty = \max\{|a|, |b|\} < \min\{|a|, |b|\}|S| \leq \|f\|_1,$$

for $|S|$ sufficiently large. From the above discussion it is clear that the choice of the norm for evaluating $\pi_R^\top - \pi_P^\top$ depends on the application.

In the light of the above discussion it is worth noting that the v -norm can be adjusted to the problem under consideration. To see this, recall that we have assumed that v is of the form $v(i) = \alpha^i$, $i \in S$, with α some unspecified constant. Let us express this dependency of v on α by writing v_α . Hence, the best bound for $|\tilde{\mu}^\top f - \mu^\top f|$ by means of the v -norm is given by the solution of the following minimization problem

$$(2.14) \quad |\tilde{\mu}^\top f - \mu^\top f| \leq \min_\alpha \|\tilde{\mu}^\top - \mu^\top\|_{v_\alpha} \|f\|_{v_\alpha}.$$

The upside of this minimization is that it trades off the effect the norm has on the reward and the measure distance. The downside is of course that the minimization itself can be rather demanding as $\|\tilde{\mu}^\top - \mu^\top\|_{v_\alpha}$ or a bound thereof typically has a complex form.

2.3. Perturbation Bounds. In perturbation analysis, D_P occurs in conjunction with a perturbation matrix $\Delta = R - P$ which has row sums equal to zero. From $\Delta(I - \Pi_P) = \Delta$ and (2.4) it follows that

$$(2.15) \quad \Delta(I - T)^{-1}(I - \Pi_P) = \Delta D_P$$

and instead of D_P for perturbation bounds it suffices to consider

$$(2.16) \quad (I - T)^{-1}(I - \Pi_P).$$

Note that due to the fact that $\Delta(I - T)^{-1}$ fails to have row sums equal to zero, the term $I - \Pi_P$ on the LHS in (2.15) cannot be disregarded. In other words, $\Delta(I - T)^{-1} \neq \Delta(I - T)^{-1}(I - \Pi_P)$, except for special cases. By simple algebra, it holds for Markov transition matrices R and P that

$$(2.17) \quad \pi_R^\top = \pi_P^\top + \pi_R^\top(R - P)D_P$$

$$(2.18) \quad = \pi_P^\top + \pi_R^\top(R - P)(I - T)^{-1}(I - \Pi_P).$$

REMARK 1. *The above formula is called update formula and allows for deriving a first perturbation bound. Using the fact that $\|\pi_R^\top\|_\infty \leq 1$, (2.18) yields*

$$\|\pi_R^\top - \pi_P^\top\|_\infty \leq \|R - P\|_\infty \|(I - T)^{-1}(I - \Pi_P)\|_\infty,$$

which provides a first perturbation bound. Put differently $\|(I-T)^{-1}(I-\Pi_P)\|_\infty$ yields a condition number.

Repeated insertion of the expression for π_R in (2.17) on RHS of (2.17), we obtain

$$(2.19) \quad \pi_R^\top = \pi_P^\top \sum_{k=0}^N ((R-P)D_P)^k + \pi_R^\top ((R-P)D_P)^{N+1}.$$

We call

$$B(R, P) = \lim_{N \rightarrow \infty} \pi_R^\top ((R-P)D_P)^N$$

the *bias matrix* provided that the limit exists. Letting N tend to infinity in (2.19) we arrive at

$$(2.20) \quad \begin{aligned} \pi_R^\top &= \pi_P^\top \sum_{k=0}^{\infty} ((R-P)D_P)^k + B(R, P) \\ &= \pi_P^\top (I - (R-P)D_P)^{-1} + B(R, P), \end{aligned}$$

provided the series exists and the bias matrix is finite. As we will explain in the following, the bias matrix is typically zero in case that R and P are unichain. The series in (2.20) already appears without the bias matrix in [32]. It has been rediscovered in [9] and extended to Markov chains on a general state-space in [15], both references study problem classes where the bias matrix is zero.

In deriving the series expansion in (2.20) we required that the stationary distribution π_R exists. As the next theorem shows, convergence of the series already implies existence of π_R . Moreover, we provide sufficient conditions for the bias matrix to be equal to the zero matrix.

THEOREM 2.3. *Let P be irreducible, aperiodic and positive recurrent. Suppose that the series in (2.20) converges to some finite limit μ^\top , i.e., let*

$$\mu^\top = \pi_P^\top (I - (R-P)D_P)^{-1}.$$

- (i) *If $\mu_i \geq 0$, for $i \in S$, then μ is a stationary distribution of R .*
- (ii) *If R is irreducible and aperiodic and there exists $i \in S$ such that $\|iR\| < 1$, then μ is the unique stationary distribution of R and $B(R, P)$ is the zero matrix.*

Proof. To see that μ is an invariant with respect to R , note that,

$$\Pi_P + (I - P)D_P = I.$$

Multiplying the above equation from the left by μ , yields

$$(2.21) \quad \pi_P^\top + \mu^\top (I - P)D_P = \mu^\top.$$

By simple algebra,

$$(2.22) \quad \begin{aligned} \mu^\top &= \pi_P^\top \sum_{k=0}^{\infty} ((R-P)D_P)^k \\ &= \pi_P^\top + \pi_P^\top \sum_{k=1}^{\infty} ((R-P)D_P)^k \\ &= \pi_P^\top + \pi_P^\top \sum_{k=0}^{\infty} ((R-P)D_P)^k (R-P)D_P \\ &= \pi_P^\top + \mu^\top (R-P)D_P. \end{aligned}$$

Subtracting (2.21) from (2.22) yields

$$\mu^\top(I - R)D_P = 0.$$

Existence of D_P implies that $D_P = (I - P + \Pi_P)^{-1} - \Pi_P$, see (2.2). Since $(I - R)\Pi_P = 0$, it holds that

$$\mu^\top(I - R)(I - P + \Pi_P)^{-1} = 0.$$

Multiplying the above equation from the right with $(I - P + \Pi_P)$ yields $\mu = \mu R$, which shows that μ is invariant to R . Further, multiplying (2.21) from the right with an appropriate column vector of ones, i.e., $\bar{1}$, shows

$$(2.23) \quad \pi_P^\top \bar{1} + \mu^\top(I - P)D_P \bar{1} = \mu^\top \bar{1} \Leftrightarrow \mu^\top \bar{1} = 1$$

since $(I - P)D_P \bar{1} = (I - \Pi_P)\bar{1} = 0$. This shows that μ sums up to 1. Provided that μ is component-wise a non-negative vector, then μ is a stationary distribution, which proves part (i).

For part (ii), note that by Corollary 1 it follows that R is positive recurrent. This together with the assumption that R is irreducible and aperiodic implies that R is ergodic and

$$(2.24) \quad \lim_{n \rightarrow \infty} R^n = \Pi_R,$$

where Π_R is a matrix with all rows equal to π_R^\top and π_R is the unique stationary distribution of R . Since all rows of Π_R are identical to π_R^\top and $\mu^\top \bar{1} = 1$, it holds that

$$(2.25) \quad \mu^\top \Pi_R = \pi_R^\top.$$

We have already shown that μ is an invariant distribution of R . This together with (2.24) and (2.25) yields

$$\mu^\top = \lim_{n \rightarrow \infty} \mu^\top R^n = \mu^\top \Pi_R = \pi_R^\top.$$

Uniqueness of the solution follows from ergodicity of R and the bias matrix is consequently the zero matrix, which concludes the proof. \square

REMARK 2. *Part (i) of Theorem 2.3 applies in case that R is a multi-chain with transient states. In this case the stationary distribution is not unique.*

The series in (2.20) can be facilitated for deriving perturbation bounds by

$$(2.26) \quad \begin{aligned} \pi_R^\top - \pi_P^\top &= \pi_P^\top \sum_{k=1}^{\infty} ((R - P)D_P)^k + B(R, P) \\ &= \pi_P^\top (R - P)D_P \sum_{k=0}^{\infty} ((R - P)D_P)^k + B(R, P) \\ (2.27) \quad &= \pi_P^\top (R - P)D_P (I - (R - P)D_P)^{-1} + B(R, P). \end{aligned}$$

Following the above line of equations, bounding $\pi_R^\top - \pi_P^\top$ requires bounding $(I - (R - P)D_P)^{-1}$. Moreover, we will show that the conditions put forward in the following lemma not only imply norm bounds for $(I - (R - P)D_P)^{-1}$ but also imply that $B(R, P)$ is the zero matrix.

LEMMA 2.4. *For any matrix norm it holds with the above notation that:*

(i) If $\|(R - P)D_P\| < 1$, then

$$\|(I - (R - P)D_P)^{-1}\| \leq \frac{1}{1 - \|(R - P)D_P\|},$$

(ii) if $\|R - P\| \|D_P\| < 1$, then

$$\|(I - (R - P)D_P)^{-1}\| \leq \frac{1}{1 - \|R - P\| \|D_P\|},$$

(iii) if $\|T\| + \|R - P\|(1 + \|\pi_P\|) < 1$, then

$$\|(I - (R - P)D_P)^{-1}\| \leq \frac{1 - \|T\|}{1 - \|T\| - \|R - P\|(1 + \|\pi_P^\top\|)}.$$

In addition, any of the conditions (i), (ii) or (iii) implies that the bias matrix equals the zero matrix.

Proof. We only provide a proof of part (iii) as the proofs of (i) and (ii) can be obtained from a similar (and simpler) line of arguments. Using the taboo kernel representation in (2.4) it holds that

$$(R - P)D_P = (R - P) \sum_{k=0}^{\infty} T^k (I - \Pi_P).$$

We have assumed that $\|T\| < 1$ and applying norms yields

$$(2.28) \quad \|(R - P)D_P\| \leq \|R - P\| \frac{1 + \|\pi_P^\top\|}{1 - \|T\|}.$$

Our condition $\|T\| + \|R - P\|(1 + \|\pi_P\|) < 1$ is equivalent to the expression on the above RHS being strictly less than 1. This implies that the Neumann series $\sum_k ((R - P)D_P)^k$ converges. Consequently $(I - (R - P)D_P)$ is invertible with norm bounded by

$$\begin{aligned} \|(I - (R - P)D_P)^{-1}\| &\leq \sum_{k=0}^{\infty} \|(R - P)D_P\|^k \\ &= \frac{1}{1 - \|(R - P)D_P\|}. \end{aligned}$$

Inserting the bound in (2.28) in the expression on the above RHS concludes the proof of the statement.

For the proof of the last part of the lemma, note that $\|\pi_R^\top ((R - P)D_P)^N\| \leq \|\pi_R^\top\| \|(R - P)D_P\|^N$, so that $\|(R - P)D_P\| < 1$ implies convergence of $\|\pi_R^\top ((R - P)D_P)^n\|$ to zero as n tends to infinity. \square

REMARK 3. *It is worth noting that $\|(R - P)D_P\| < 1$ typically fails in case R is a multi-chain.*

Note that

$$\|(R - P)D_P\| \leq \|R - P\| \|D_P\| \leq \frac{\|R - P\|(1 + \|\pi_P\|)}{1 - \|T\|}$$

implies that the bounds put forward in Lemma 2.4 are increasingly limited in their applicability, while the evaluation of the bounds becomes simpler. In fact, computing

$\|(R - P)D_P\|$ is often not feasible as D_P is either not known in closed form or is prohibitively complex in general, see [12, 16, 26]. For the presented Markov chains in Example 1 the D_P is known in explicit form. For this type of problems it makes sense to apply the norm bounds put forward in Lemma 2.4 to (2.27). More specifically, assuming that $\|(R - P)D_P\| < 1$ yields

$$(2.29) \quad \|\pi_R^\top - \pi_P^\top\| \leq \frac{\|\pi_P^\top(R - P)D_P\|}{1 - \|(R - P)D_P\|},$$

which we will call the *direct bound*.

REMARK 4. *The bound in (2.29) has the following nice feature. Let P and R be two Markov chains with $P \neq R$ but with the same stationary distribution. Then, (2.29) detects this and yields the correct value 0, whereas condition number type bounds yield a non-zero bound.*

The next bound can serve as alternative in case D_P is difficult to find. It follows from replacing $(R - P)D_P$ in (2.29) with the taboo kernel representation and bounding the result via (2.28). Specifically, this leads to

$$(2.30) \quad \begin{aligned} \|\pi_R^\top - \pi_P^\top\| &\leq \|\pi_P^\top\| \|R - P\| \frac{1 + \|\pi_P^\top\|}{1 - \|T\|} \frac{1 - \|T\|}{1 - \|T\| - \|R - P\|(1 + \|\pi_P^\top\|)} \\ &= \|\pi_P^\top\| \|R - P\| \frac{1 + \|\pi_P^\top\|}{1 - \|T\| - \|R - P\|(1 + \|\pi_P^\top\|)}, \end{aligned}$$

provided that $\|T\| + \|R - P\|(1 + \|\pi_P^\top\|) < 1$. The bound put forward in (2.30) is called *Strong Stability Bound* (SSB) in the literature [20]. For applications of SSB, we refer to [1, 2, 3, 7, 8, 27].

Based on the SSB we get the robust sensitivity bound for π_P stated in Lemma 2.5.

LEMMA 2.5. *Provided that $\|T\| < 1$, it holds that*

$$\frac{d\|\pi_P\|}{d\|P\|} \leq \frac{\|\pi_P^\top\|(1 + \|\pi_P^\top\|)}{1 - \|T\|}.$$

Proof. Provided that $\|T\| < 1$ we may take $\|R - P\|$ sufficiently small such that (2.30) holds. Dividing the inequality by $\|R - P\|$ and letting $\|R - P\|$ tend to zero proves the claim. \square

An obvious improvement of the bound in (2.30) is to replace $\|\pi_P^\top\| \|R - P\|$ by $\|\pi_P^\top(R - P)\|$; see Remark 4.

While P and π_P are fixed, and T offering in practice only limited flexibility, R is a free variable of the perturbation bound. Essentially, the direct bound and SSB only apply if R is not too far away from P , i.e., if $\|R - P\|$ is small. This is the major drawback of this type of perturbation bound compared to the condition numbers. To overcome this drawback, we may scale the perturbation such that the perturbation bounds do apply. To see this, consider the scaled model in (1.5), where the static perturbation is replaced by a scaled one, i.e., we perturb P by $\theta(R - P)$ and denote the resulting transition matrix by $P(\theta)$. Now, θ can be chosen such that the norm bounds apply to $\theta\|R - P\|$. For example, the condition on the applicability for SBB in (2.30) translates to

$$\|T\| + \theta\|R - P\|(1 + \|\pi_P^\top\|) < 1 \quad \text{iff} \quad 0 \leq \theta < \frac{1 - \|T\|}{\|R - P\|(1 + \|\pi_P^\top\|)}.$$

We call the upper bound for θ on the RHS above the *domain of SBB with respect to R* .

In the following we take an alternative route for obtaining a perturbation bound. Starting point is (2.17) but other than for deriving (2.20) we now only perform the insertion operation K times, leading to

$$(2.31) \quad \pi_{P(\theta)}^\top = \pi_P^\top \sum_{k=0}^K (\theta(R-P)D_P)^k + \pi_{P(\theta)}^\top (\theta(R-P)D_P)^{K+1}.$$

For $K \geq 1$, equation (2.31) yields the following bound:

$$(2.32) \quad \|\pi_{P(\theta)}^\top - \pi_P^\top\| \leq \left\| \pi_P^\top \sum_{k=1}^K (\theta(R-P)D_P)^k \right\| + \|\pi_{P(\theta)}^\top (\theta(R-P)D_P)^{K+1}\|.$$

Obviously, $\pi_{P(\theta)}^\top$ is not known and for the actual bound and we use the fact that

$$\begin{aligned} \|\pi_{P(\theta)}^\top (\theta(R-P)D_P)^{K+1}\| &\leq \|\pi_{P(\theta)}^\top\| \|(\theta(R-P)D_P)^{K+1}\| \\ &\leq c_{\|\cdot\|} \|(\theta(R-P)D_P)^{K+1}\|, \end{aligned}$$

where we define the norm dependent upper bound $c_{\|\cdot\|}$ for $\|\pi_{P(\theta)}^\top\|$ as follows

$$(2.33) \quad c_{\|\cdot\|} = \sup_{Q \in \mathbb{P}(S)} \|\pi_Q^\top\|.$$

In case the 1-norm (resp. sup-norm) is applied to $\pi_{P(\theta)}^\top$ we thus have

$$(2.34) \quad \|\pi_{P(\theta)}^\top (\theta(R-P)D_P)^{K+1}\| \leq \|(\theta(R-P)D_P)^{K+1}\|.$$

For the general v -norm, a bound $c_{\|\cdot\|}$ can be obtained from (2.11). Note that a trivial bound for $c_{\|\cdot\|}$ is given by α^{n-1} .

The *series expansion perturbation bound of order K* (SEB(K)) is now introduced by

$$(2.35) \quad \|\pi_{P(\theta)}^\top - \pi_P^\top\| \leq \left\| \pi_P^\top \sum_{k=1}^K (\theta(R-P)D_P)^k \right\| + c_{\|\cdot\|} \|(\theta(R-P)D_P)^{K+1}\|,$$

where $c_{\|\cdot\|}$ is as defined in (2.33).

REMARK 5. *Note that we may bound (2.35) as follows*

$$(2.36) \quad \|\pi_{P(\theta)}^\top - \pi_P^\top\| \leq \sum_{k=1}^K \|\pi_P^\top ((R-P)D_P)^k\| \|\theta^k\| + c_{\|\cdot\|} \|((R-P)D_P)^{K+1}\| \|\theta^{K+1}\|,$$

so that the polynomial terms only have to be calculated once and can be used for evaluating the bound for different values of θ . This allows for fast computation and memory efficiency but, due to the additional bounding, the numerical quality of the bound decreases.

From

$$\|((R-P)D_P)^{K+1}\| \leq \|(R-P)D_P\|^{K+1}$$

it follows that the series in (2.20) converges for $P(\theta) = P + \theta(R - P)$ at least for $\theta < (\|(R - P)D_P\|)^{-1}$. Hence, for θ sufficiently small

$$(2.37) \quad \pi_P^\top \sum_{k=0}^K (\theta(R - P)D_P)^k$$

provides an approximation of $\pi_{P(\theta)}$, where the error is bounded by some constant times $\theta^{K+1}\|(R - P)D_P\|^{K+1}$. The series put forward in (2.37) is called *series expansion approximation* (SEA) of order K . Letting K tend to infinity in (2.37) we obtain for θ sufficiently small that

$$\pi_{P(\theta)}^\top = \pi_P^\top \sum_{k=0}^{\infty} \theta^k ((R - P)D_P)^k.$$

Note that the above series expansion implies that $\pi_{P(\theta)}$ tends to π_P as θ tends to zero. For more details about SEA see [17, 16].

To test the performance of the different bounds in the scaled perturbation setting (i.e., (1.4)) we will investigate the relative error of the perturbation bounds. Clearly, a better bound results in a smaller relative error. Consider a condition number bound for $\|\pi_{P(\theta)}^\top - \pi_P^\top\|$. Then, the relative error inferred by using $\theta\|R - P\|\kappa$ rather than $\|\pi_{P(\theta)}^\top - \pi_P^\top\|$ is given by

$$\frac{\theta\|R - P\|\kappa - \|\pi_{P(\theta)}^\top - \pi_P^\top\|}{\|\pi_{P(\theta)}^\top - \pi_P^\top\|} = \frac{\theta\|R - P\|\kappa}{\|\pi_{P(\theta)}^\top - \pi_P^\top\|} - 1.$$

Note that this relative error is by the definition of $\text{CNB} \geq 0$. The following theorem analyses the relative error of the discussed bounds. It shows that in general the relative error of a condition number bound converges for $\theta \downarrow 0$ to a finite non-zero value. This means that even for a small perturbation this bound has a significant relative error. The same holds true for the SSB, while the SE-based bounds have the desirable property that the relative error vanishes. Moreover, the rate of convergence of the relative error of SEB can be explicitly computed.

THEOREM 2.6 (Relative Errors). *Let $\|\pi_{P(\theta)}^\top - \pi_P^\top\| > 0$, for all $\theta \in (0, 1]$.*

(i) *The relative error of the condition number bound (CNB) is given by*

$$\eta_{\text{CNB}}(\theta) = \frac{\|R - P\|\kappa}{\|\pi_{P(\theta)}^\top (R - P)D_P\|} - 1,$$

and it holds that

$$\lim_{\theta \downarrow 0} \eta_{\text{CNB}}(\theta) = \frac{\|R - P\|\kappa}{\|\pi_P^\top (R - P)D_P\|} - 1 \geq 0,$$

where equality is only reached in the special case when $\|R - P\|\kappa$ equals $\|\pi_P^\top (R - P)D_P\|$.

(ii) *Provided that $\|T\| + \theta\|R - P\|(1 + \|\pi_P^\top\|) < 1$, the relative error of the strong stability bound (SSB) is given by*

$$\eta_{\text{SSB}}(\theta) = \frac{\|R - P\|\|\pi_P^\top\|(1 + \|\pi_P^\top\|)}{\|\pi_{P(\theta)}^\top (R - P)D_P\|(1 - \|T\| - \theta\|R - P\|(1 + \|\pi_P^\top\|))} - 1,$$

and it holds that

$$\lim_{\theta \downarrow 0} \eta_{\text{SSB}}(\theta) = \frac{\|R - P\| \|\pi_P^\top\| (1 + \|\pi_P^\top\|)}{\|\pi_P^\top (R - P) D_P\| (1 - \|T\|)} - 1 \geq 0,$$

where equality is only reached in the special case when the nominator equals the denominator in the fraction.

(iii) Provided that $\theta \|(R - P) D_P\| < 1$, the relative error of the direct bound (DB) is given by

$$\eta_{\text{DB}}(\theta) = \frac{\frac{\|\pi_P^\top (R - P) D_P\|}{1 - \theta \|(R - P) D_P\|}}{\|\pi_{P(\theta)}^\top (R - P) D_P\|} - 1,$$

and it holds that $\lim_{\theta \downarrow 0} \eta_{\text{DB}}(\theta) = 0$.

(iv) Provided that $\theta \|(R - P) D_P\| < 1$, the relative error of the series expansion bound of order $K \geq 1$ (i.e., $\text{SEB}(K)$) is given by

$$\eta_{\text{SEB}(K)}(\theta) = \frac{2c_{|\cdot|} \|((R - P) D_P)^{K+1}\| \theta^K}{\|\pi_{P(\theta)}^\top (R - P) D_P\|},$$

and it holds that $\eta_{\text{SEB}(K)}(\theta)$ is of order $O(\theta^{K-1})$.

Proof. All relative error expressions follow by simply inserting the different bound and using the result that

$$(2.38) \quad \pi_{P(\theta)}^\top - \pi_P^\top = \theta \pi_{P(\theta)}^\top (R - P) D_P,$$

in the denominator (see also (2.17)). E.g., for the CNB it holds that

$$\eta_{\text{CNB}}(\theta) = \frac{\theta \|R - P\| \kappa}{\|\pi_{P(\theta)}^\top - \pi_P^\top\|} - 1 = \frac{\|R - P\| \kappa}{\|\pi_{P(\theta)}^\top (R - P) D_P\|} - 1,$$

where we simplified the expression in the second equation. For the limit, we use that $\pi_{P(\theta)}$ tends to π_P as θ tends to zero, which follows from (2.35) for $K = 1$.

We now turn to the computing the relative error for the K -th order SE. Following (2.35) we can write

$$(2.39) \quad \eta_{\text{SEB}(K)}(\theta) = \frac{\overbrace{\|\pi_P^\top \sum_{k=1}^K (\theta (R - P) D_P)^k\|}^{=:H} + c_{|\cdot|} \|(\theta (R - P) D_P)^{K+1}\|}{\|\theta \pi_{P(\theta)}^\top (R - P) D_P\|} - 1.$$

For H it holds that

$$H = \|\pi_P^\top \sum_{k=0}^{K-1} (\theta (R - P) D_P)^k \theta (R - P) D_P\|.$$

After some algebra,

$$H = \left\| \pi_P^\top \sum_{k=0}^{\infty} (\theta (R - P) D_P)^k [I - (\theta (R - P) D_P)^K] \theta (R - P) D_P \right\|$$

and using condition $\|\theta(R - P)D_P\| < 1$ together with Theorem 2.3 we arrive at

$$H = \left\| \pi_{P(\theta)}^\top [I - (\theta(R - P)D_P)^K] \theta(R - P)D_P \right\|,$$

which can be straightforwardly bounded by

$$H \leq \|\pi_{P(\theta)}^\top \theta(R - P)D_P\| + c_{\|\cdot\|} \|\theta(R - P)D_P\|^{k+1}.$$

Inserting the above bound for H into (2.39) yields for the relative error

$$(2.40) \quad \eta_{\text{SEB}(K)}(\theta) \leq \frac{2c_{\|\cdot\|} \|((R - P)D_P)^{K+1}\|}{\|\pi_{P(\theta)}^\top (R - P)D_P\|} \theta^K.$$

We now turn to establishing the rate of convergence of $\eta_{\text{SEB}(K)}(\theta)$. First, note that the nominator on the above RHS is of order $O(\theta^K)$. We now turn to the denominator. Evoking the update formula (2.20), it follows that

$$\pi_{P(\theta)}^\top (R - P)D_P = \pi_{P(\theta)}^\top - \pi_P^\top = \sum_{k=1}^{\infty} \theta^k ((R - P)D_P)^k,$$

which shows that $\pi_{P(\theta)}^\top (R - P)D_P$ can be written as power series with leading term $\theta(R - P)D_P$. Finiteness of the matrices and their norms therefore implies that $\|\pi_{P(\theta)}^\top (R - P)D_P\|$ is of order $O(\theta)$. Hence, $\eta_{\text{SEB}(K)}(\theta)$ is of order $O(\theta^{K-1})$. \square

REMARK 6. *Theorem 2.6 illustrates a conceptual limitation of condition number bounds since the relative error of a condition number bound fails to tend to zero as θ tends to zero. The same holds for SSB.*

As Theorem 2.6 shows, perturbation bounds have the intrinsic drawback that in general the relative error does not vanish for small perturbations. For the condition number bounds their applicability is questionable due to the fact that a simple scaling with respect to the perturbation size is assumed whereas SEB(K) shows that the dependence of $\pi_{P(\theta)}$ on θ is non-linear. SSB, even though not a linear type of bound, suffers from problem that the domain of applicability is so small that the non-linearity of the functional form of SSB does not come into play. SEB(K) is of a polynomial type and has an asymptotic relative error with specified rate of convergence.

For an illustration of Theorem 2.6 we generated two random transition matrices P and R with 40 states. The random generation is done by drawing random numbers from $(0, 1)$ and normalizing the rows so that they sum up to 1. Then we considered the perturbation bounds from Theorem 2.6 on the interval $\theta \in (0, 1]$ together with the true perturbation effect $\|\pi_{P(\theta)}^\top - \pi_P^\top\|$. The results can be found in Figure 1. Figure 1 shows that in this experiment all bounds, except for CNB, are similar in performance on the interval $\theta \in [0, 0.1]$. For $\theta > 0.1$ there arises a difference in performance, where the SEB of order $K = 3$ performs best. DB performs similar to SEB(1) on the interval $\theta \in (0, 0.3]$ but for $\theta > 0.3$ SEB(1) outperforms DB. This simple example illustrates that the CNB is apparently too general to be competitive compared to the other bounds. The differences become more apparent if we look at the relative errors for the different bounds plotted in Figure 2. The results for SSB are not plotted because the condition in part (ii) of Lemma 2.4 is not met.

REMARK 7. *Provided that θ_0 exists such that $\theta_0 \|(R - P)D_P\| < 1$, then*

$$\eta_{\text{SEB}(K)}(\theta) = O(\theta_0^{K-1}),$$

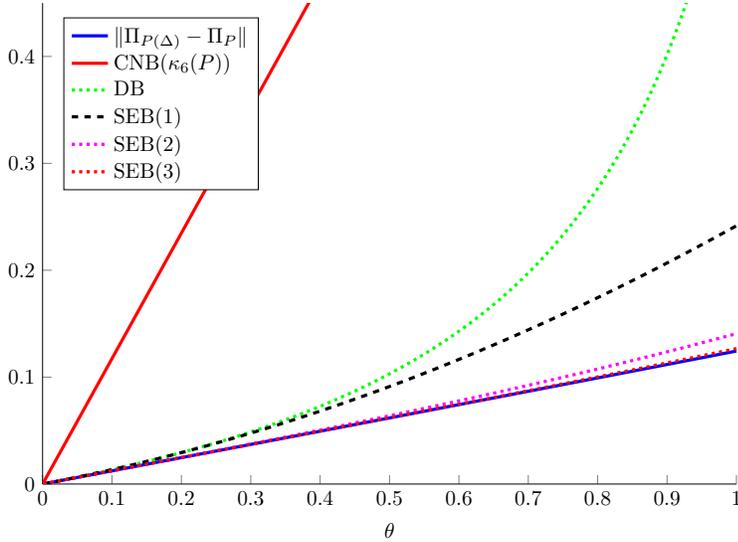


FIG. 1. Perturbation bounds for $\|\pi_{P(\theta)}^\top - \pi_P^\top\|$ with $\theta \in (0, 1]$, where $P(\theta) = (1 - \theta)P + \theta R$ for randomly generated P and R consisting of 40 states.

for $0 \leq \theta \leq \theta_0$.

We conclude this section by presenting an interesting result for stability theory.

COROLLARY 2.7. *Consider the model $P(\theta) = (1 - \theta)P + \theta R$, $\theta \in [0, 1]$, with P aperiodic, irreducible and positive recurrent. If*

$$\theta < \frac{1 - \|_i P\|}{\|R - P\|},$$

then $P(\theta)$ has a unique stationary distribution.

Proof. Note that $P(\theta)$ is aperiodic and irreducible for $\theta \in [0, 1]$. It remains to be shown that $P(\theta)$ is positive recurrent. By computation,

$$\begin{aligned} \|_i(P(\theta))\| &= \|_i((1 - \theta)P + \theta R)\| \\ &\leq \|_i P + \theta(R - P)\| \\ &\leq \|_i P\| + \theta\|R - P\|. \end{aligned}$$

Hence, provided that θ satisfies $\|_i P\| + \theta\|R - P\| < 1$, it follows $\|_i(P(\theta))\| < 1$ and by Corollary 1 we conclude that $P(\theta)$ is positive recurrent. Solving θ out of $\|_i P\| + \theta\|R - P\| < 1$ concludes the proof. \square

REMARK 8. *Note that from Corollary 2.7 it follows that if condition (ii) for the SSB in Theorem 2.6 is satisfied, then $P(\theta)$ is stable, i.e., has a unique stationary distribution.*

Kartashov established in [20] a result similar to Theorem 2.3. It is worth noting that Kartashov didn't provide a lower bound for the region of stability as detailed in Corollary 2.7 together with Remark 8.

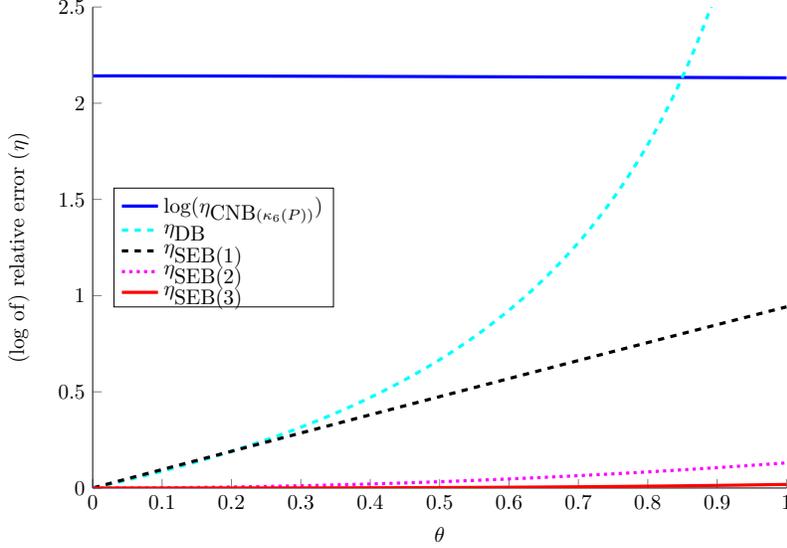


FIG. 2. Relative errors of the perturbation bounds for $\|\pi_{P(\theta)}^\top - \pi_P^\top\|$ with $\theta \in (0, 1]$, where $P(\theta) = (1 - \theta)P + \theta R$ for randomly generated P and R consisting of 40 states.

3. Explicit Perturbation Bounds for the Small Network. In this section we explicitly compute the bounds from Theorem 2.6, i.e., CNB, SSB, DB and $\text{SEB}(K)$ (for $K = 0, 1$), for the Small Network from Example 2. The following convex combination is considered

$$P(\theta) = (1 - \theta) \underbrace{\begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}}_{=P^s} + \theta \underbrace{\begin{pmatrix} 1-\tilde{p} & \tilde{p} \\ \tilde{q} & 1-\tilde{q} \end{pmatrix}}_{:=\tilde{P}^s}.$$

We are interested perturbing $P(0)$ by choosing $\theta > 0$. Note that for the difference in Markov transition matrices it holds

$$P(\theta) - P(0) = \theta(P(\theta) - P^s) = \theta \begin{pmatrix} p - \tilde{p} & \tilde{p} - p \\ \tilde{q} - q & q - \tilde{q} \end{pmatrix}.$$

which gives

$$\|P(\theta) - P(0)\|_v = \theta(1 + \alpha) \max \left\{ |p - \tilde{p}|, \frac{1}{\alpha} |q - \tilde{q}| \right\}.$$

In the following the explicit perturbation bounds are presented for the v -norm. Using (2.17) in the calculation for CNB we get

$$\|\pi_{P(\theta)}^\top - \pi_{P^s}^\top\|_v \leq \|\pi_{P(\theta)}^\top\|_v \|P(\theta) - P^s\|_v \|D_{P^s}\|_v.$$

It holds that (see also Example 2)

$$\|\pi_{P(\theta)}^\top\|_v \leq \alpha \quad \text{and} \quad \|D_{P^s}\|_v = \frac{1 + \alpha}{(p + q)^2} \max \left\{ p, \frac{q}{\alpha} \right\}$$

so that we obtain for the CNB

$$\|\pi_{P(\theta)}^\top - \pi_{P^s}^\top\|_v \leq \theta \left(\frac{1+\alpha}{p+q} \right)^2 \max\{\alpha|p-\tilde{p}|, |q-\tilde{q}|\} \max\left\{p, \frac{q}{\alpha}\right\}.$$

In the general framework of CNB given in (1.1) it holds that $\kappa = \frac{1+\alpha}{(p+q)^2} \max\{\alpha p, q\}$ for this example.

For the SSB we compute

$$\|\pi_{P^s}^\top\|_v = \frac{q+p\alpha}{p+q}.$$

Next, the individual terms in (2.30) have to be computed. Here, we make use of the taboo kernel bound as provided in Example 2 after removing the first column. SSB can only be provided for small perturbations, i.e., small values of θ . More specifically, provided that

$$\theta < \frac{1 - \max\{\alpha p, 1 - q\}}{\left(1 + \frac{q+p\alpha}{p+q}\right) (1 + \alpha) \max\left\{|p - \tilde{p}|, \frac{1}{\alpha}|q - \tilde{q}|\right\}},$$

the SSB bound for $\|\pi_{P(\theta)}^\top - \pi_{P^s}^\top\|_v$ is given by

$$\frac{\left(\frac{q+p\alpha}{p+q}\right) \left(1 + \frac{q+p\alpha}{p+q}\right) \theta (1 + \alpha) \max\left\{|p - \tilde{p}|, \frac{1}{\alpha}|q - \tilde{q}|\right\}}{1 - \min\{\max\{\alpha p, 1 - q\}, \max\{\alpha(1 - p), q\}\} - \left(1 + \frac{q+p\alpha}{p+q}\right) \theta (1 + \alpha) \max\left\{|p - \tilde{p}|, \frac{1}{\alpha}|q - \tilde{q}|\right\}}.$$

For example, letting $\alpha = 1$, which is possible, see Lemma 2.1, yields the simplified expression

$$\frac{4\theta \max\{|p - \tilde{p}|, |q - \tilde{q}|\}}{1 - \min\{\max\{p, 1 - q\}, \max\{1 - p, q\}\} - 4\theta \max\{|p - \tilde{p}|, |q - \tilde{q}|\}}$$

for SSB. By inspection of above, it is obvious that SSB behaves poorly for p and q close to one or close to zero as in this case the norm of the taboo kernel approaches one.

Calculations show that DB leads to

$$\|\pi_{P(\theta)}^\top - \pi_{P^s}^\top\|_v \leq \frac{\theta|p\tilde{q} - \tilde{p}q|(1 + \alpha)}{(p + q) \left(p + 1 - \theta(1 + \alpha) \max\left\{|p - \tilde{p}|, \frac{|q - \tilde{q}|}{\alpha}\right\}\right)}$$

under the assumption that

$$\theta < \frac{p + 1}{(1 + \alpha) \max\left\{|p - \tilde{p}|, \frac{|q - \tilde{q}|}{\alpha}\right\}}.$$

For $\text{SEB}(K)$ with $K = 0$ it holds

$$\|\pi_{P(\theta)}^\top - \pi_{P^s}^\top\|_v \leq \frac{\theta(1 + \alpha)}{p + q} \max\{\alpha|p - \tilde{p}|, |q - \tilde{q}|\}$$

of which the construction is similar to CNB but with the difference that CNB requires an additional bounding on $\|(P(\theta) - P^s)D_{P^s}\|_v$ to obtain $\|(P(\theta) - P^s)\|_v \|D_{P^s}\|_v$,

which stems from the fact that $\|(P(\theta) - P^s)D_{P^s}\|_v \leq \|(P(\theta) - P^s)\|_v \|D_{P^s}\|_v$. More specifically, CNB is by factor

$$\frac{\text{CNB}}{\text{SEB}(0)} = \frac{1 + \alpha}{p + q} \max\left\{p, \frac{q}{\alpha}\right\} \geq 1$$

larger than $\text{SEB}(0)$. In case $\alpha = 1$ this factor is $2 \max\{p, q\}/(p + q)$, which is greater than 1 for $p \neq q$. When α is chosen to be $\gg 1$ this factor is likely to grow linearly in α . This illustrates that, although being more general, CNB loses on quality in contrast to $\text{SEB}(0)$ since it does not utilize the contraction property of $(P(\theta) - P^s)D_{P^s}$.

After similar calculations it can be shown that $\text{SEB}(K)$ with $K = 1$ results in

$$\|\pi_{P(\theta)}^\top - \pi_{P^s}^\top\|_v \leq \frac{\theta(1 + \alpha)}{(p + q)^2} (|p\tilde{q} - \tilde{p}q| + \theta|p - \tilde{p} + q - \tilde{q}| \max\{\alpha|p - \tilde{p}|, |q - \tilde{q}|\}).$$

4. An Elaborate Perturbation Analysis of a Queueing System. To illustrate the application of perturbation bounds in a setting where the deviation matrix is not available we discuss in this section the M/G/1 queue with breakdowns. In addition, we consider the finite version of the queue, i.e., the M/G/1/N queue with breakdowns and we illustrate $\text{SEB}(K)$. The breakdown model will have the special feature that we perturb the system with no breakdowns by an unstable chain modeling a pure birth process.

The basic model of the M/G/1 queue with breakdowns is introduced in Section 4.1 and in Section 4.2 a discussion of the literature is provided. The perturbation bounds for both models are presented in Section 4.3 and Section 4.4, respectively.

4.1. The Basic Model. Consider a single server queue. Customers arrive to the queue according to a Poisson- λ -arrival process. Service times are identically distributed with mean $1/\mu$ and we denote the service time distribution by $S(x)$. Throughout this section we assume that $\lambda/\mu < 1$. At the beginning of each service, there is a probability θ that the server breaks down (and the customer is send back to the queue) and enters a repair state, the length of which is exponentially distributed with rate r and which is independent of everything else, and with probability $(1 - \theta)$ the server is operational and serves the customer (if any, according to FCFS). The only points in time where a possible server breakdown can occur is right at the beginning of a service. This system is modeled by the jump chain embedded at service completions and completions of a repair, and it has state space $S = \{0, 1, \dots\}$. The transition probabilities from $i \in S$ to $j \in S$, denoted as $P_\theta(i, j)$, are given as follows:

For $i = 0$, the process jumps to $j \geq 0$ if a customer arrives and the server is operational, or if during the service of this customer there are j additional arrivals. This probability is given by

$$(1 - \theta) \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dS(x).$$

Alternatively, a customer arrives to the empty queue and the server breaks down at service initiation and during the repair time of the server there are $j - 1$ additional arrivals, so that at the end of the repair time there are in total j customers at the server. This probability is given by

$$\theta \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-1}}{(j-1)!} r e^{-rx} dx = \theta \frac{r}{\lambda + r} \left(\frac{\lambda}{\lambda + r} \right)^{j-1},$$

for $j \geq 1$ and zero for $j = 0$, where we make use of the convention that $0! = 1$. Combining these results, for $i = 0$, we arrive at

$$P_\theta(0, j) = (1 - \theta) \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dS(x) + \theta \frac{r}{\lambda + r} \left(\frac{\lambda}{\lambda + r} \right)^{j-1} \mathbf{1}_{j \geq 1}.$$

For $i \geq 1$, the process jumps to state $j \geq 0$ if the server remains operationally, so that service of the subsequent customer in the queue may begin, and during the service of this customer there are $j - i + 1 \geq 0$ additional arrivals. This probability is given by

$$(1 - \theta) \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} dS(x).$$

Alternatively, there is a server breakdown and during the exponential repair time there are $j - i \geq 0$ arrivals from the outside. This probability is given by

$$\theta \frac{r}{\lambda + r} \left(\frac{\lambda}{\lambda + r} \right)^{j-i}.$$

Combining these results, we arrive at

$$(4.1) \quad P_\theta(i, j) = (1 - \theta) \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} dS(x) + \theta \frac{r}{\lambda + r} \left(\frac{\lambda}{\lambda + r} \right)^{j-i} \mathbf{1}_{j \geq i},$$

for $1 \leq i$ and $i - 1 \leq j$. All other entries of P_θ are set to zero.

Observe that for $\theta = 1$, P_1 models a pure birth process and the queue is not stable, whereas P_0 models a stable M/G/1 queue with no breakdowns. The kernel P_θ is given through the convex combination $\theta P_1 + (1 - \theta)P_0$ of the two kernels.

4.2. Discussion of Literature. Since the pioneering work of Thiruvengadam [34] and Avi-Itzhak and Naor [5], there has been a considerable interest in the study of queues with server breakdowns, see for example [10, 28, 35] and references therein. However, the majority of results is expressed in terms of systems of equations the solution of which is rather challenging, or have solutions which are not easily interpretable in practice. For instance, Baccelli and Znati [6] provide the generating function of the number of customers in the M/G/1 system with dependent breakdowns. Also, results are given in terms of the inverse of Laplace transforms, see, e.g., [6], which require numerical inversion of solving a given system. To overcome these difficulties, approximation methods are used where the complex (real) system is replaced by one which is “close” to it in some sense but which has a simpler in structure (resp. components) and for which analytical results are available.

4.3. The Infinite Capacity M/G/1 Queue with Breakdowns. In this section the M/G/1 Queue with Breakdowns is considered. Note that SSB is the only bound applicable as the size of the state-space is infinite and the deviation matrix is not known in explicit form. As next we provide auxiliary results for obtaining the overall SSB. Recall that P_0 is the transition kernel of the embedded jump chain of an M/G/1 queue and that $T =_0(P_0)$.

For the taboo kernel T it holds that

$$\begin{aligned}
\|T\|_v &= \sup_{i \geq 0} \frac{1}{\alpha^i} \sum_{j \geq 1} \alpha^j \left| \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} dS(x) \right| \mathbf{1}_{j-i+1 \geq 0} \\
&= \sup_{i \geq 0} \frac{1}{\alpha^i} \sum_{j \geq 1} \alpha^j \left| \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} dS(x) \right| \mathbf{1}_{j \geq i-1} \\
&= \sup_{i \geq 0} \frac{1}{\alpha^i} \sum_{j \geq \max(i-1, 1)} \alpha^j \left| \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dS(x) \right|
\end{aligned}$$

For $i = 0, 1$,

$$\begin{aligned}
\sup_{0 \leq i \leq 1} \frac{1}{\alpha^i} \sum_{j \geq \max(i-1, 1)} \alpha^j \left| \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dS(x) \right| &\leq \sum_{j \geq 1} \alpha^j \left| \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dS(x) \right| \\
&= \sum_{j \geq 1} \int_0^\infty e^{-\lambda x} \frac{(\lambda \alpha x)^j}{j!} dS(x) \\
&= \int_0^\infty e^{-\lambda x} \sum_{j \geq 1} \frac{(\lambda \alpha x)^j}{j!} dS(x) \\
&= \int_0^\infty e^{-\lambda x} (e^{\lambda \alpha x} - 1) dS(x) \\
&= \int_0^\infty e^{-\lambda(1-\alpha)x} dS(x) - \int_0^\infty e^{-\lambda x} dS(x),
\end{aligned}$$

and for $i > 1$

$$\begin{aligned}
&\sup_{i \geq 2} \frac{1}{\alpha^i} \sum_{j \geq \max(i-1, 1)} \alpha^{j-1} \left| \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dS(x) \right| \\
&= \sup_{i \geq 2} \frac{1}{\alpha^i} \sum_{j \geq i-1} \alpha^{j-1} \left| \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dS(x) \right| \\
&\leq \sup_{i \geq 2} \frac{1}{\alpha^i} \sum_{j \geq 1} \alpha^{j-1} \left| \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dS(x) \right| \\
&= \frac{1}{\alpha^3} \int_0^\infty e^{-\lambda x} \sum_{j \geq 0} \frac{(\lambda \alpha x)^j}{j!} dS(x) - \frac{1}{\alpha^3} \int_0^\infty e^{-\lambda x} dS(x) \\
&\leq \frac{1}{\alpha^3} \left(\int_0^\infty e^{-\lambda(1-\alpha)x} dS(x) - \int_0^\infty e^{-\lambda x} dS(x) \right).
\end{aligned}$$

where we have used the condition Denoting by $S^*(z)$ the Laplace-Stieltjes transform of $S(x)$ and using the fact that $\alpha \geq 1$ and that α satisfies (??) we arrive at

$$\|T\| = \|_0(P_0)\|_v \leq b_1(\alpha) := S^*(\lambda(1-\alpha)) - S^*(\lambda),$$

provided that α is such that

$$(4.2) \quad S^*(\lambda(1-\alpha)) < \infty.$$

Furthermore, following the same line of argument one obtains

$$\|\pi_0^\top\|_v \leq b_2(\alpha) := \frac{1}{1 - b_1(\beta)} \left(\pi_0(0) \int e^{-\lambda x} dS(x) + \pi_0(1) \int e^{-\lambda x} (\lambda x) dS(x) \right).$$

We now turn to computing a bound for $\|P_1 - P_0\|_v$. For $i = 0$:

$$\begin{aligned} & \sum_{j \geq 0} \alpha^j |P_1(0, j) - P_0(0, j)| \\ &= \sum_{j \geq 0} \alpha^j \left| \frac{r}{r + \lambda} \left(\frac{\lambda}{\lambda + r} \right)^{j-1} 1_{j \geq 1} - \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} dS(x) \right| \\ &= \int_0^\infty e^{-\lambda x} dS(x) + \sum_{j \geq 0} \alpha^j \left| \frac{r}{r + \lambda} \left(\frac{\lambda}{\lambda + r} \right)^j - \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j+1}}{(j+1)!} dS(x) \right|. \end{aligned}$$

For $i \geq 1$:

$$\begin{aligned} & \frac{1}{\alpha^i} \sum_{j \geq 0} \alpha^j |P_1(i, j) - P_0(i, j)| \\ &= \frac{1}{\alpha^i} \sum_{j \geq 0} \alpha^j \left| \frac{r}{r + \lambda} \left(\frac{\lambda}{\lambda + r} \right)^{j-i} 1_{j \geq i} - \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} dS(x) \right| 1_{j-i+1 \geq 0} \\ &= \int_0^\infty e^{-\lambda x} dS(dx) + \frac{1}{\alpha^i} \sum_{j \geq i} \alpha^j \left| \frac{r}{r + \lambda} \left(\frac{\lambda}{\lambda + r} \right)^{j-i} - \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} dS(x) \right| \\ &= \int_0^\infty e^{-\lambda x} dS(dx) + \frac{1}{\alpha^i} \sum_{j \geq 0} \alpha^j \left| \frac{r}{r + \lambda} \left(\frac{\lambda}{\lambda + r} \right)^j - \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j+1}}{(j+1)!} dS(x) \right|. \end{aligned}$$

Combining the above results we let

$$b_3(\alpha) := \int_0^\infty e^{-\lambda x} dS(x) + \sum_{j \geq 0} \alpha^j \left| \frac{r}{r + \lambda} \left(\frac{\lambda}{\lambda + r} \right)^j - \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^{j+1}}{(j+1)!} dS(x) \right|$$

and obtain

$$\|P_1 - P_0\|_v \leq b_3(\alpha).$$

Inserting the above bounds into (2.30) we obtain as SSB

$$\|\pi_\theta^\top - \pi_0^\top\|_v \leq b_2(\alpha) \frac{\theta(1 + b_2(\alpha))b_3(\alpha)}{1 - b_1(\alpha) - \theta(1 + b_2(\alpha))b_3(\alpha)},$$

provided that

$$\theta < \frac{1 - b_1(\alpha)}{(1 + b_2(\alpha))b_3(\alpha)}$$

and $1 \leq \alpha \leq \min(1/\lambda, z_\lambda)$, where z_λ denotes the right point of the domain of the values for α such that $S^*(\lambda(1 - \alpha))$ is finite (the case $z_\lambda = \infty$ is not excluded).

EXAMPLE 4. *If the service times are exponentially distributed with rate μ it holds that*

$$S^*(\lambda(1 - \alpha)) = \frac{\mu}{\mu + \lambda(1 - \alpha)}$$

and $z_\lambda = \frac{\mu+\lambda}{\lambda} - \epsilon$, for $\epsilon > 0$. The above bounds can now be explicitly computed:

$$b_1(\alpha) := \frac{\mu}{\mu + \lambda(1 - \alpha)} - \frac{\mu}{\mu + \lambda} = \frac{\lambda\mu\alpha}{(\mu + \lambda)(\mu + \lambda(1 - \alpha))},$$

$$b_2(\alpha) := \frac{1}{1 - b_1(\alpha)}(1 + \lambda r)e^{-\lambda r}.$$

and

$$b_3(\alpha) = \frac{\mu}{\lambda + \mu} + \sum_{j \geq 0} \alpha^j \left| \frac{r}{r + \lambda} \left(\frac{\lambda}{\lambda + r} \right)^j - \left(\frac{\lambda}{\lambda + \mu} \right)^{j+1} \frac{1}{j + 1} \right|.$$

Note that in case $\mu = r$, $b_3(\alpha)$ simplifies to

$$b_3(\alpha) = \frac{\mu}{\lambda + \mu} + \sum_{j \geq 0} \alpha^j \frac{j}{j + 1} \left(\frac{\lambda}{\lambda + \mu} \right)^j \leq \frac{\mu}{\lambda + \mu} + \frac{1}{1 - \frac{\alpha\lambda}{\lambda + \mu}} = \frac{\mu}{\lambda + \mu} + \frac{\lambda + \mu}{(1 - \alpha)\lambda + \mu},$$

provided that $\alpha < \frac{\lambda + \mu}{\lambda}$.

In the following, we let $\lambda = 0.5$, $\mu = 1$, $r = 1$ and $f(s) = 0$ for $s \leq 10$ and $f(s) = 1$ for $s > 10$, i.e., we are interested in the probability of having more than 10 customers at the queue in stationary regime, i.e.,

$$\|f\|_v = \frac{1}{\alpha^{11}}.$$

For ease of computation we assume that the service times are exponentially distributed.

We are now able to apply the bound provided in Lemma 2.2 to $|\pi_\theta f - \pi_0 f|$ in combination with the above SSB, where we let θ vary from 0 to 0.01, see Figure 3. The minimization with respect to α in (2.14) has been solved numerically. As can be seen from Figure 3, SSB provides qualitative insight rather than numerically satisfying approximations.

Recall that $T =_0(P_0)$ and, by Remark 8, applicability of SSB implies stability of the system with breakdowns. SSB can thus be used as means of establishing a lower bound for the domain of stability of the queue with breakdowns. More precisely, by Example 4, for $\mu = r = 1$ condition

$$\|T\| \leq b_1(\alpha) < 1$$

implies

$$\alpha \leq \frac{(\mu + \lambda)^2}{(2\mu + \lambda)\lambda},$$

which yields for the numerical setting of our example

$$\alpha \leq \frac{9}{5}.$$

In accordance with Corollary 2.7, a lower bound for the region of stability of $P(\theta)$ is

$$\frac{1 - \|T\|}{\|P_1 - P_0\|} \geq \max_{1 \leq \alpha \leq 9/5} \frac{(\mu + \lambda)(\mu + \lambda(1 - \alpha)) - \mu\lambda\alpha}{\mu(\lambda(1 - \alpha) + \mu) + (\lambda + \mu)^2},$$

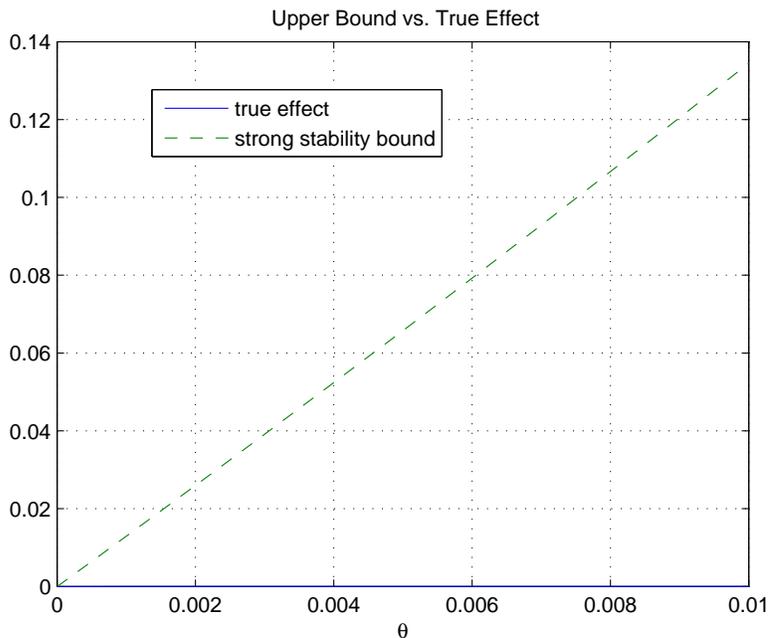


FIG. 3. The true change in probability of more than 10 customers in the system vs. the strong stability bound.

where we used the bounds provided in Example 4. For the numerical values of the example we obtain

$$\max_{1 \leq \alpha \leq 9/5} \frac{9 - 5\alpha}{18 - 2\alpha} = \frac{1}{4},$$

where the maximum is attained at $\alpha = 1$. Hence, the system remains stable for a breakdown probability up to $\approx 1/4$.

In the following section, we will show that the series expansion bound yields numerically better bounds. This comes, however, at the price of restricting the analysis to a finite version of model.

4.4. The M/G/1/N Queue with Breakdowns. In this section a M/G/1/N queue is considered of finite size N (where N is not too large). In this case the state space is $S = \{0, 1, \dots, N\}$. D_θ (short for D_{P_θ}) and π_θ (short for π_{P_θ}) can be easily computed numerically. In this case, SEB can be used for numerical computations. We illustrate the series expansion bound with some numerical examples. We choose $N = 50$ as the maximum number of jobs in the system. Furthermore, similar as in the previous section we let $\lambda = 0.5$, $\mu = 1$, and $r = 1$. Like in the previous section, we assume that service times are exponentially distributed.

REMARK 9. *Note that for large N the mean queue length of the finite system is (almost) identical to that of the infinite one. In this case one could use the strong stability bounds for approximate performance evaluation rather than computing the SEB explicitly.*

We compute SEB for the v -norm with $\alpha = 1$. We have to check the condition put forward in (iv) of Theorem 2.6 numerically. For our numerical setting we obtain

$\|(P_1 - P_0)D_0\|_v = 8$ which implies $\theta\|(P_1 - P_0)D_0\|_v < 1$ for $0 \leq \theta \leq \theta_0 < 1/8$. In the following we choose $\theta_0 = 0.1$.

In Figure 4 we plot the relative absolute error of $\text{SEB}(K)$ for $K = 8, 9$ and 10 , for the probability of having more than 10 customers in the finite capacity systems. More specifically, we bound $|\pi_\theta^\top f - \pi_0^\top f|$ for $\theta \in [0, \theta_0]$, with $\theta_0 = 0.1$, using $\text{SEB}(K)$, where $f(s) = 1$ if $s > 10$ and zero otherwise. It thus holds that $\|f\|_v = 1$. In line with Lemma 2.2, we obtain the bound

$$|\pi_\theta^\top f - \pi_0^\top f| \leq \text{SEB}(K).$$

We plot in Figure 4 the absolute relative error, given by

$$\frac{\text{SEB}(K)}{|\pi_\theta^\top f - \pi_0^\top f|},$$

for $K = 8, 9, 10$ and $\theta \in [0, 0.1]$.

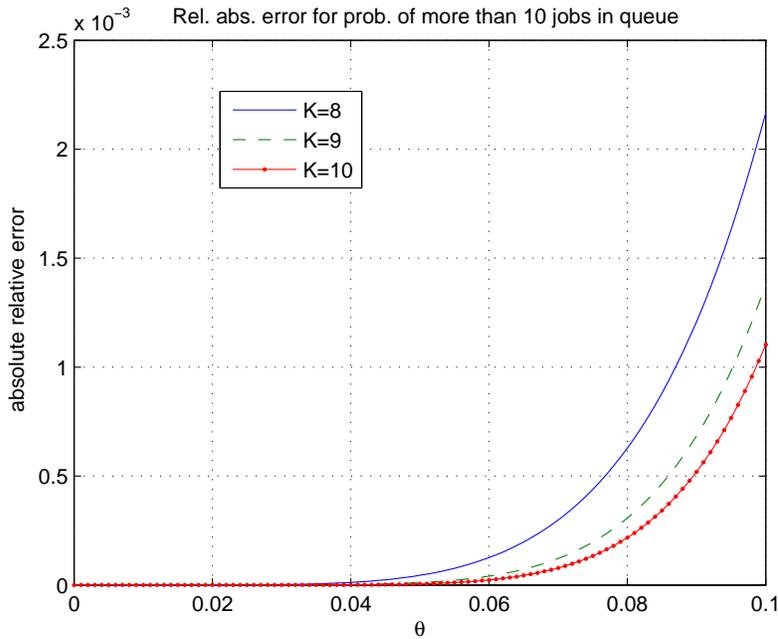


FIG. 4. The relative absolute error for approximating the $|\pi_\theta^\top f - \pi_0^\top f|$ with $\text{SEB}(K)$ with $K = 8, 9$ and 10 .

4.5. Discussion of Results. In this section we discussed numerical approximations for the single server queue with breakdowns. SSB has the advantage of providing bounds for infinite queues, unfortunately, the numerical quality of the bounds is rather poor. In light of Theorem 2.6, this comes as no surprise. SEB proved to be numerically very efficient for the model but required that a finite queue is studied. There is, however, an interesting link between the two approaches as the techniques developed for SSB lend themselves to establish lower bounds of convergence for series expansions.

5. Conclusion. Perturbation bounds for Markov chains have been intensively studied in the literature. Condition number bounds are attractive as they provide uniform perturbation bounds. However, due to their simple structure they cannot capture the true non-linear dependence of the stationary distribution on the Markov kernel. Unfortunately, they also yield rather poor results in general for small perturbations as the relative error fails to tend to zero as the size of the perturbation tends to zero. In addition, the bounds behave poorly for large systems. We explained the poor behavior of the condition number bounds as they are actually bounding the robust sensitivity and thus behave almost by definition poorly when applied to a specific perturbation. SSB is the only bound applicable bound for the infinite state space case. It is a non-linear expression in the size of the perturbation but it can typically only be applied to (very) small perturbations and thus its non-linear aspect is of no effect. We provided a new bound, i.e., SEB, which yielded good results and with provably vanishing relative error when the perturbation goes to zero. A realistic example from queueing theory was used to illustrate the potential use of perturbation bounds.

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