Chapter 5

Manifestation of the Gouy phase in strongly focused, radially polarized beams

This Chapter is based on

- X. Pang and T.D. Visser,
  “Manifestation of the Gouy phase in strongly focused, radially polarized beams,”
  *Optics Express* vol. 21, pp. 8331-8341 (2013).

Abstract
The Gouy phase, sometimes called the focal phase anomaly, is the curious effect that in the vicinity of its focus a diffracted field, compared to a nondiffracted, converging spherical wave of the same frequency, undergoes a rapid phase change by an amount of $\pi$. We theoretically investigate the phase behavior and the polarization ellipse of a strongly focused, radially polarized beam. We find that the significant variation of the state of polarization in the focal region, is a manifestation of the different Gouy phases that the two electric field components undergo.
5.1 Introduction

The phase anomaly is a measure of how the phase of a monochromatic, focused wave field differs from that of a non-diffracted, converging spherical wave of the same frequency. Since its first description by L.G. Gouy in the 1890s [Gouy, 1890; Gouy, 1891], his namesake phase has been observed under a wide variety of circumstances. Recently investigated systems range from vortex beams [Baumann et al., 2009; Philip et al., 2012] to fields of surface plasmon polaritions [Zhu et al., 2007]. Surprisingly many different explanations for the physical origin of this remarkable effect have been suggested (see [Visser and Wolf, 2010] and the references therein). Because of its crucial role in many applications such as mode conversion [Beijersbergen et al., 1993], coherence tomography [Lamouche et al., 2004], the tuning of the resonance frequency of laser cavities [Klaassen et al., 2004], and interference microscopy (Chapter 4 of this thesis), the Gouy phase continues to attract attention.

When a beam of light is focused by a high-aperture system, the usual scalar formalism no longer suffices, and an analysis of the Gouy phase must then take the vector nature of the field into account. This has recently been done for strongly focused, linearly polarized beams (see Chapter 2 of this thesis). It was found that the Gouy phases of the three Cartesian components of the electric field exhibit quite different behaviors. Another example which requires a vectorial description is the focusing of radially polarized beams [Youngworth and Brown, 2000; Martínez-Herrero and Mejías, 2012]. Because of their intriguing properties, such as a relatively small focal spot size [Dorn et al., 2003], these beams are widely used in, for example, the probing of the dipole moment of individual molecules [Novotny et al., 2001], high-resolution microscopy [Sheppard and Choudhury, 2004], trapping of strongly scattering particles [Zhan, 2004b; Nieminen et al., 2008] and in dark-field imaging [Biss et al., 2006]. A review is presented in [Brown, 2011].

A first indication of the complicated phase behavior of focused, radially polarized beams was the observation that their wave spacing near focus is highly irregular [Visser and Foley, 2005]. This was followed by a study of the Gouy phase of the longitudinal component of the electric field vector at the focal plane [Chen et al., 2007]. In the present chapter, the Gouy phase of the total electric field vector, consisting of a radial
and longitudinal longitudinal component, is examined in the entire focal region. It is found that the strong changes in the shape and orientation of the polarization ellipse near focus is a consequence of the different Gouy phases that these two components undergo.

5.2 Focused, radially polarized fields

Consider an aplanatic focusing system $L$ of focal length $f$ with a semi-aperture angle $\alpha$. The geometrical focus is indicated by $O$ and is taken to be the origin of a Cartesian coordinate system (see Fig. 5.1). A monochromatic, radially polarized beam with angular frequency $\omega$ is incident upon the system. The electric and magnetic fields at time $t$ at position $r$ are given by the expressions

$$E(r, t) = \text{Re} \left[ e(r) \exp(-i\omega t) \right],$$

$$H(r, t) = \text{Re} \left[ h(r) \exp(-i\omega t) \right],$$

respectively, where $\text{Re}$ denotes the real part. Such a field may be generated, for example, by the superposition of two, mutually orthogonally polarized, Hermite-Gaussian beams. If we assume that the entrance plane of the focusing system coincides with the waist plane of the beam, then the longitudinal component $e_z$ and the radial component $e_\rho$ of the electric field at a point $P = (\rho, z)$ in the focal region are given by the equations [VISSER
5.2. Focused, radially polarized fields

AND FOLEY, 2005

\[
e_z(\rho, z) = -ikf \int_0^\alpha l(\theta) \sin^2 \theta \cos^{1/2} \theta \\
\times e^{ikz \cos \theta} J_0(k \rho \sin \theta) \, d\theta, \quad (5.3)
\]

\[
e_\rho(\rho, z) = -kf \int_0^\alpha l(\theta) \sin \theta \cos^{3/2} \theta \\
\times e^{ikz \cos \theta} J_1(k \rho \sin \theta) \, d\theta, \quad (5.4)
\]

where \( J_i \) is the Bessel function of the first kind of order \( i \) and \( k = \omega/c \), with \( c \) the speed of light in vacuum, is the wavenumber associated with frequency \( \omega \). Furthermore, \( l(\theta) \) denotes the angular amplitude function

\[
l(\theta) = f \sin \theta \exp[-f^2 \sin^2 \theta/\omega_0^2], \quad (5.5)
\]

where \( \omega_0 \) is the spot size of the beam in the waist plane. Note that since the incident electric field has no azimuthal component and the configuration is invariant with respect to rotations around the \( z \)-axis, there is no azimuthal component of the electric field in the focal region. The position of an observation point \( P \) may be indicated by the dimensionless Lommel variables \( u \) and \( v \) [RICHARDS AND WOLF, 1959], namely

\[
u = k \rho \sin \alpha.
\] (5.7)

Eqs. (5.3) and (5.4) can then be rewritten as

\[
e_z(u, v) = -ikf^2 \int_0^\alpha \sin^3 \theta \cos^{1/2} \theta e^{-\beta^2 \sin^2 \theta} \\
\times e^{iu \cos \theta/\sin^2 \alpha} \frac{J_0}{\sin \alpha} \left( \frac{v \sin \theta}{\sin \alpha} \right) \, d\theta, \quad (5.8)
\]

\[
e_\rho(u, v) = -kf^2 \int_0^\alpha \sin^2 \theta \cos^{3/2} \theta e^{-\beta^2 \sin^2 \theta} \\
\times e^{iu \cos \theta/\sin^2 \alpha} \frac{J_1}{\sin \alpha} \left( \frac{v \sin \theta}{\sin \alpha} \right) \, d\theta, \quad (5.9)
\]

\footnote{This was first derived in [YOUNG WORTH AND BROWN, 2000], but we adopt the notation of [VISSE R AND FOLEY, 2005].}
where

$$\beta = \frac{f}{\omega_0},$$  \hspace{1cm} (5.10)

which denotes the ratio of the focal length of the system and the spot size of the beam in the waist plane.

It follows from Eqs. (5.8) and (5.9) that the field obeys the following symmetry relations:

$$e_z(-u, v) = -e_z^*(u, v), \hspace{1cm} (5.11)$$
$$e_\rho(-u, v) = e_\rho^*(u, v). \hspace{1cm} (5.12)$$

### 5.3 Two Gouy phases

The Gouy phase $\delta$ is defined as the difference between the actual phase of the field and that of a (non-diffracted) spherical wave converging to the focus in the half-space $z < 0$ and diverging from it in the half-space $z > 0$ ([Born and Wolf, 1999, Sec. 8.8, Eq. (48)]). For each individual component of the electric field we therefore define a Gouy phase as

$$\delta_z(u, v) = \text{arg}[e_z(u, v)] - \text{sign}(u)kR, \hspace{1cm} (5.13)$$
$$\delta_\rho(u, v) = \text{arg}[e_\rho(u, v)] - \text{sign}(u)kR, \hspace{1cm} (5.14)$$

where $R$ is the distance from the observation point to the geometrical focus, i.e.

$$kR = k\sqrt{z^2 + \rho^2} = \frac{1}{\sin \alpha} \sqrt{\frac{u^2}{\sin^2 \alpha} + v^2}, \hspace{1cm} (5.15)$$

and $\text{sign}(x)$ denotes the sign function

$$\text{sign}(x) = \begin{cases} 
-1 & \text{if } x < 0, \\
1 & \text{if } x > 0. 
\end{cases} \hspace{1cm} (5.16)$$

For the longitudinal field component $e_z$, one finds from Eqs. (5.8), (5.11) and (5.13) that the Gouy phase at two points that are symmetrically located with respect to the focus satisfies the relation

$$\delta_z(-u, v) + \delta_z(u, v) = -\pi \text{ (mod } 2\pi). \hspace{1cm} (5.17)$$
At the focus we have

$$\delta_z(0,0) = -\pi/2 \pmod{2\pi}. \quad (5.18)$$

For the radial field component $e_\rho$, it follows from Eqs. (5.9), (5.12) and (5.14) that the Gouy phase satisfies the symmetry relations

$$\delta_\rho(-u,v) + \delta_\rho(u,v) = 0 \pmod{2\pi}. \quad (5.19)$$

Even though $e_\rho = 0$ when $v = 0$, it is useful to study the behavior of $\delta_\rho$ along a tilted ray through focus; for such a ray $v \propto |u|$. Using the fact that for small arguments $J_n(x) \sim x^n$ [Abramowitz and Stegun, 1965, p. 360] we find from Eq. (5.9) that along such a ray $e_\rho \sim -|u|$. Hence

$$\lim_{u \downarrow 0} \delta_\rho(u,v) = \lim_{u \uparrow 0} \delta_\rho(u,v) = \pi \pmod{2\pi}. \quad (5.20)$$

It is seen from Eqs. (5.8) and (5.9) that the electric field components are characterized by two parameters, namely the semi-aperture angle $\alpha$, and the beam-size parameter $\beta$. These two parameters have a different effect on the Gouy phase behavior as we will now demonstrate.

On the central axis of the system ($v = 0$) only the longitudinal field component $e_z$ is non-zero. The Gouy phase pertaining to this component, $\delta_z$, is shown in Fig. 5.2 for various values of the semi-aperture angle $\alpha$. It is seen that the phase change of $e_z$ decreases as $\alpha$ increases. Unlike the $\pi$ phase jump of the longitudinal component in linearly polarized fields (see Chap. 2 of this thesis), the Gouy phase here is continuous at focus. Note that the longitudinal coordinate $u$ is dependent on the value of the semi-aperture angle $\alpha$ [See Eq. (5.6)]. In Fig. 5.3 the Gouy phase $\delta_z$ is depicted for selected values of the beam-size parameter $\beta$. For a decreasing beam waist-size ($\omega_0$) the Gouy phase decreases as well. In these two figures, the negative or positive slope of the Gouy phase means that the wavefront spacings can be smaller or bigger than $\lambda$. This has been discussed by [Visser and Foley, 2005].
Chapter 5. Gouy phase of a radially polarized beams

Figure 5.2: The Gouy phase $\delta_z(u,v = 0)$ along the optical axis ($v = 0$) of the electric field component $e_z$ for selected values of the semi-aperture angle $\alpha$ (blue curve: $\alpha = 40^\circ$, red curve: $\alpha = 50^\circ$, olive curve: $\alpha = 60^\circ$). The beam-size parameter $\beta = 3$.

Figure 5.3: The Gouy phase $\delta_z(u,v = 0)$ along the optical axis ($v = 0$) of the electric field component $e_z$ for selected values of the beam-size parameter $\beta = f/\omega_0$ (green curve: $\beta = 1$, blue curve: $\beta = 2$, red curve: $\beta = 3$, olive curve: $\beta = 4$). The semi-aperture angle $\alpha = 60^\circ$.

When $v \neq 0$, it follows from Eqs. (5.8) and (5.9) that both the lon-
The Gouy phase and the state of polarization

5.4 The Gouy phase and the state of polarization

It is convenient to characterize the state of polarization of a two-dimensional field by the four Stokes parameters ([BORN AND WOLF, 1999], Sec. 1.4). For a beam propagating in the z-direction, these parameters are defined in terms of $e_x$ and $e_y$. For a focused, radially polarized fields the two non-zero components of the electric field are $e_z$ and $e_\rho$. It is natural, therefore, to define the Stokes parameters in this case in terms of these components rather than $e_x$ and $e_y$ [SCHOONOVER AND VISser, 2006; MARTíNEz-HERRERO...
We thus define

\[ S_0 = |e_z|^2 + |e_\rho|^2, \] (5.21)

\[ S_1 = |e_z|^2 - |e_\rho|^2, \] (5.22)

\[ S_2 = 2|e_z||e_\rho|\cos\delta, \] (5.23)

\[ S_3 = 2|e_z||e_\rho|\sin\delta, \] (5.24)

where \( \delta = \arg[e_z] - \arg[e_\rho] = \delta_z - \delta_\rho. \) The normalized forms \( s_1 = S_1/S_0, \)
\( s_2 = S_2/S_0, \) and \( s_3 = S_3/S_0, \) can be represented as a point on the Poincaré sphere [GBUR, 2011, p. 316], as shown in Fig. 5.5. On the northern hemisphere \( (s_3 > 0) \), the polarization is right-handed (clockwise), whereas on the southern hemisphere it is left-handed (counter-clockwise). On both poles \( (s_3 = \pm 1) \), the polarization is circular and on the equator \( (s_3 = 0) \), it is linear.

![Figure 5.5: The Poincaré sphere with Cartesian axes \((s_1, s_2, s_3)\) adapted for focused, radially polarized fields.](image)

Along a ray through focus, which makes an angle \( \theta \) with the \( z \)-axis,
we have

\[ v = |u| \tan \theta / \sin \alpha. \quad (5.25) \]

From Eqs. (5.11) and (5.12) it immediately follows that

\[ |e_z(-u, v)| = |e_z(u, v)|, \quad (5.26) \]
\[ |e_\rho(-u, v)| = |e_\rho(u, v)|. \quad (5.27) \]

These two relations are illustrated in Fig. 5.6. They also imply that the first Stokes parameter \( S_0 \) is an even function in \( u \). Using Eqs. (5.17) and (5.19), it is seen that

\[ [\delta_z(u, v) - \delta_\rho(u, v)] + [\delta_z(-u, v) - \delta_\rho(-u, v)] = \pi \quad (mod \ 2\pi), \quad (5.28) \]

for the quantity \( \delta \), which is defined below Eq. (5.24), this implies that \( \cos[\delta(-u, v)] = -\cos[\delta(u, v)] \) and \( \sin[\delta(-u, v)] = \sin[\delta(u, v)] \). Thus we find the following symmetry relations for the normalized Stokes parameters along a ray through focus:

\[ s_1(-u, v) = s_1(u, v), \quad (5.29) \]
\[ s_2(-u, v) = -s_2(u, v), \quad (5.30) \]
\[ s_3(-u, v) = s_3(u, v). \quad (5.31) \]

An example is presented in Fig. 5.7. It is seen that \( S_0, S_1 \) and \( S_3 \) are even, whereas \( S_2 \) is odd.
Figure 5.6: The normalized moduli of the longitudinal component $e_z$ and that of the radial component $e_\rho$ of the electric field along an oblique ray under an angle $\theta = 35^\circ$ with the $z$-axis. Here we have chosen $\alpha = 40^\circ$ and $\beta = 1$.

The polarization ellipse may be characterized by two angular parameters (see Fig. 5.8). One is the orientation angle, $\psi$ ($0 \leq \psi < \pi$), which is the angle between the $z$-axis and the major axis of the polarization ellipse. The other is the ellipticity angle, $\chi$ ($-\pi/4 \leq \chi < \pi/4$). $|\tan \chi|$ represents the ratio of the axes of the ellipse. The values $\pm \pi/4$ correspond to circular polarization; whereas the value 0 indicates linear polarization. The sign of $\chi$ distinguishes the two senses of handedness, i.e., it is right-handed when $\chi > 0$, and left-handed when $\chi < 0$, see [Born and Wolf, 1999, Sec. 1.4]. The two angular parameters can be expressed in terms of the normalized Stokes parameters, as

$$\psi = \frac{1}{2} \arctan \left( \frac{s_2}{s_1} \right), \quad (5.32)$$

$$\chi = \frac{1}{2} \arcsin(s_3). \quad (5.33)$$
5.4. The Gouy phase and the state of polarization

Figure 5.7: The Stokes parameters along an oblique ray through focus which makes an angle $\theta = 35^\circ$ with the $z$-axis. Here $\alpha = 40^\circ$ and $\beta = 1$.

Figure 5.8: Defining the angles $\psi$ and $\chi$ of a polarization ellipse.

From the symmetry relations of $s_1$, $s_2$ and $s_3$, it is seen that

\begin{align}
\psi(-u, v) &= \pi - \psi(u, v), \\
\chi(-u, v) &= \chi(u, v).
\end{align}

(5.34) (5.35)

Two kinds of polarization singularities can occur. When the polarization ellipse is circular, the orientation angle $\psi$ is undefined. This happens at so-called $C$-points. When the polarization is linear, the handedness is
undefined. This occurs at so-called $L$-points. When a system parameter, such as the semi-aperture angle $\alpha$, is varied in a continuous manner, these polarization singularities can be created or annihilated. This has been described in [Schoonover and Visser, 2006; Diehl et al., 2006].

The curves of the orientation angle $\psi$ and the ellipticity angle $\chi$ along an oblique ray under angle $\theta = 35^\circ$ are displayed in Figs. [5.9] and [5.10]. In Fig. [5.9] it is seen that the orientation angle of the ellipse oscillates somewhat along the ray. Also, a $C$-point is seen near $u = \pm 1.2$, where the orientation angle $\psi$ is singular. To the left of the $C$-point at $u = -1.2$, the polarization ellipse is slightly larger in the $\rho$-direction than it is in the $z$-direction. This situation is reversed to the right of that $C$-point. This coincides with a $\pi/2$ jump of the angle $\psi$. In Fig. [5.10] these $C$-points occur when the ellipticity angle $\chi$ takes on the value $\pi/4$. When $\chi$ equals $0$, an $L$-point occurs, which happens near points such as $u = \pm 2.9$.

In Fig. [5.11] the polarization ellipse is shown for different positions along an oblique ray. The ellipses at $(u, v)$ and $(u, -v)$ have the same ellipticity and handedness, whereas their orientations are mirror-symmetric. The changes in the polarization ellipse are closely related to the two Gouy phases, as we will now discuss. When $u = -4$, it is seen from Fig. [5.4] that $\delta = \delta_z - \delta_\rho < 0$, according to Eq. (5.24) the polarization is then counter-clockwise which corresponds to a point on the southern half of the Poincaré sphere. Near the point $u = -2.87$, $\delta_z = \delta_\rho$, and hence the field is linearly polarized with its handedness undefined, corresponding to a point on the equator. In the vicinity of the focus, the Gouy phase difference, $\delta_z - \delta_\rho$, is approximately $\pi/2$ (see Fig. [5.4]) and when $u = -1.2$ the moduli of the two components attain the same magnitude (see Fig. [5.6]), therefore the field there is circularly polarized which corresponds to a point on the North pole. The field is linearly polarized at focus due to the zero amplitude of the field component $e_\rho$. We also find that from $u = -4$ to $u = -1.2$ the handedness of the polarization changes from counter-clockwise, to undefined, to clockwise.
5.4. The Gouy phase and the state of polarization

\[ \psi(u, \theta = 35^\circ) \]

Figure 5.9: The orientation angle \( \psi \) of the polarization ellipse along an oblique ray through focus under an angle \( \theta = 35^\circ \). Here we have chosen \( \alpha = 40^\circ \) and \( \beta = 1 \).

\[ \chi(u,v) \]

Figure 5.10: The ellipticity angle \( \chi \) of the polarization ellipse along an oblique ray through focus under an angle \( \theta = 35^\circ \). Here we have chosen \( \alpha = 40^\circ \) and \( \beta = 1 \).

It is seen from Figs. 5.12, 5.13, and 5.14 that along different oblique rays through focus, the polarization ellipse goes through different shapes and states of handedness. This behavior mirrors the different Gouy phases.
along these rays. In Fig. 5.12 (with $\theta = 10^\circ$) the handedness is clockwise at all observation points. This means that the Stokes parameter $s_3 > 0$, i.e. $\delta_z - \delta_\rho > 0$. From Eq. (5.24), we see that this implies that the relative change of the two Gouy phases is limited along this ray. This is also the case for $\theta = 20^\circ$, as can be seen from Fig. 5.13. In that case, however, the ellipticity is considerably larger. If the obliquity angle $\theta$ is further increased to $30^\circ$ (see Fig. 5.14), the polarization ellipses becomes even narrower. In addition, the handedness evolves from counter-clockwise to clockwise, reflecting the fact that $\delta_z - \delta_\rho$ changes sign along the ray. Finally, the change in the orientation angle of the ellipses is seen to decrease significantly when the angle $\theta$ is increased.

![Polarization Ellipse](image)

Figure 5.11: Polarization ellipse of the field at selected points along an oblique ray through focus. The ray is under an angle $\theta = 35^\circ$. Also, $\alpha = 40^\circ$ and $\beta = 1$. 
Figure 5.12:  Polarization ellipse of the field at selected points along an oblique ray through focus. The ray is under an angle $\theta = 10^\circ$. Also, $\alpha = 40^\circ$ and $\beta = 1$.

Figure 5.13:  Polarization ellipse of the field at selected points along an oblique ray through focus. The ray is under an angle $\theta = 20^\circ$. Also, $\alpha = 40^\circ$ and $\beta = 1$. 
5.5 Conclusions

We have analyzed the phase behavior of strongly focused, radially polarized fields. We found that the Gouy phase of the two components of the electric field are quite different, and have different symmetries. Our results show that the semi-aperture angle $\alpha$ and the beam-size parameter $\beta$ can both influence the Gouy phase. If we follow the polarization ellipse along a tilted ray through focus, it is seen to “tumble”, i.e., it changes its orientation, its shape and handedness. This behavior is due to the different Gouy phases that the two components of the electric field undergo.
5.5. Conclusions