Summary

Random fractals and scaling limits in percolation

In this thesis we study the scaling limits of various percolation models as well as certain random fractal models. We briefly introduce the models and state the main results of this thesis.

In Chapter 3 we consider Bernoulli site percolation on the triangular lattice: each site of the triangular lattice is declared open with probability p and closed with probability $1 - p$, independently of other sites. Interface curves are the boundaries between open and closed clusters (where we identified each site of the triangular lattice with its corresponding hexagon in the dual lattice, i.e. the hexagonal lattice) and we study the distribution of interface curves as the mesh size δ tends to 0. The limit distribution is the scaling limit we are interested in. We let $p = p_\delta$ depend on δ in such a way that $p_\delta \to 1/2$ as $\delta \to 0$ and show that, depending on the choice of p_δ, there are three possibilities for the scaling limit and we give a qualitative description of these scaling limits.

In Chapter 4 we turn our attention to random Voronoi percolation. This model is obtained from a Poisson process with intensity λ in the following way. Declare each Poisson point x black (white) with probability p $(1 - p)$, independently of other points, and color its associated Voronoi cell $V(x)$ correspondingly. Again, we are interested in the behavior of the distribution of interface curves in the limit as $\lambda \to \infty$. Based on the assumption of a natural RSW result for random Voronoi percolation we derive the existence of a scaling limit for random Voronoi percolation.

In Chapter 5 we study Mandelbrot fractal percolation which is defined as follows. Consider the unit cube $[0,1]^d$, for $d \geq 2$. Let $N \geq 2$ be an integer and partition the unit cube into N^d subcubes of equal size. Each subcube is retained with probability p and discarded otherwise, independently of other subcubes. This yields a random subset $D_p^1 \subset [0,1]^d$. We repeat the previous procedure in every retained subcube at all smaller scales. This yields an infinite decreasing sequence $(D_p^n)_{n\geq 1}$ of (compact) random subsets of $[0,1]^d$. We are interested in its limit set $D_p = \bigcap_{n=1}^{\infty} D_p^n$. The set D_p can be decomposed in two disjoints sets: the set D_p^c of connected components and D_p^d, the set of totally disconnected points. We show that Hausdorff dimension of D_p^c, the set of connected components, is strictly smaller than the Hausdorff dimensions of D_p. Since Hausdorff dimension can be interpreted as a measure for "fractality" this result implies that the dust set is more fractal than the set of connected components. In addition, we show in dimension $d = 2$ with a scaling limit approach that the set of connected components is a union of non-trivial Hölder continuous curves.

In Chapter 6 we study two modified versions of Mandelbrot's fractal percolation model. An equivalent construction of the Mandelbrot fractal percolation model with parameter p is as follows. We start again by dividing the unit cube $[0,1]^d$ in N^d equal cubes with side length $1/N$. Draw a number R from a binomially distributed random variable with parameters N^d and p. Next, choose R subcubes (out of N^d possible subcubes) in a uniform way and call these cubes retained. Repeat this procedure in every retained cube at every smaller scale. It follows from standard probabilistic arguments that the limit set obtained in this way has the same distribution as D_p. Now, we modify this construction by taking R deterministic, e.g. $R = k$ for an integer k, instead of random. The other parts in the construction remain unchanged: we choose k retained subcubes in a uniform way and repeat this procedure in every retained subcube. Let $k_c(N,d)$ be the critical value, such that for $k \geq k_c(N,d)$ the unit cube is crossed with positive probability by D_k. Let $\mathbb{L}^d$ be the d-dimensional lattice with vertex set $\mathbb{Z}^d$ and with edge set given by the adjacency relation: $x \sim y$ if and only if $|x_i - y_i| \leq 1$ for all i and $x_i = y_i$ for at least one value of i. For $d = 2$, $\mathbb{L}^d$ equals just the square lattice. We prove that $k_c(N,d) \to p_c(d)$, as $N \to \infty$, where $p_c(d)$ is the critical probability for site percolation on $\mathbb{L}^d$.

The second model we study is called fat fractal percolation, because of the analogy with the fat Cantor set. The fat fractal percolation model is obtained as follows from Mandelbrot fractal percolation. Instead of a fixed percolation parameter p we retain at iteration step n a cube with probability p_n, where the sequence $\{p_n\}_{n\geq 1}$ is non-decreasing and satisfies $\prod_{n=1}^{\infty} p_n > 0$. Denote the limit set of fat fractal percolation with D_{fat}. Again, we can seperate D_{fat} in two disjoint sets: the set of totally disconnected points D^d_{fat} and the set of connected components D^c_{fat}. Let λ denote d-dimensional Lebesgue measure. We prove that either the set of totally disconnected points has positive Lebesgue measure or the set of connected components, but never simultaneously.