Summary

Representations of Erdős spaces by homeomorphism groups and by lower semi-continuous functions on product spaces

The two spaces that form the core of this thesis are Erdős space, which we denote by $\mathcal{E}$, and complete Erdős space, which we denote by $\mathcal{E}_c$. Both spaces were introduced by Erdős in 1940. Erdős space is the space of all sequences of rational numbers in $\ell^2$, the Hilbert space of square summable real sequences. Complete Erdős space is the space of all sequences in $\ell^2$, every coordinate of which is a point in the convergent sequence $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. Erdős proved that both $\mathcal{E}$ and $\mathcal{E}_c$ are one dimensional, yet totally disconnected spaces. Moreover, it is easy to see that $\mathcal{E}$ and $\mathcal{E}_c$ are homeomorphic to their own squares, that is, $\mathcal{E}$ is homeomorphic to $\mathcal{E}^2$ and $\mathcal{E}_c$ is homeomorphic to $\mathcal{E}_c^2$. This means that $\dim \mathcal{E} = \dim \mathcal{E}^2 = 1$, which makes this space an important example in dimension theory. Of course, we have a similar situation for $\mathcal{E}_c$. The spaces $\mathcal{E}$, $\mathcal{E}_c$, and also the countable infinite product of $\mathcal{E}_c$ were topologically characterized by Dijkstra and van Mill.

As will become clear when reading this thesis, $\mathcal{E}$ and $\mathcal{E}_c$ appear in many different situations. For example, in Chapter 2 our main result is Theorem 2.4.1, which states that certain homeomorphism groups of $n$-dimensional Sierpiński carpets for $n \neq 3$ turn out to be homeomorphic to $\mathcal{E}$. The proof of this result is based on a similar theorem for Menger manifolds by Dijkstra and van Mill. The key step there is the use of one of their topological characterizations of $\mathcal{E}$, which is a deep result. Moreover, we heavily use Dijkstra’s result that there are closed imbeddings of $\mathcal{E}_c$ in the homeomorphism group of $n$-dimensional Sierpiński carpets if $n \neq 3$. This amount of heavy machinery needed to
prove Theorem 2.4.1 makes it the main result of this thesis. The rea-
son for throwing the reader immediately into the deep end of the pool
like this is that the other chapters deal with various generalizations of
known constructions of $\mathcal{E}$ and $\mathcal{E}_c$; in Chapter 4 we even generalize the
construction of $\mathcal{E}_c$ to a nonseparable setting. We think it is more con-
venient to begin with a ‘pure’ construction of $\mathcal{E}$, rather than starting
with generalized constructions.

In Chapter 3 we will proceed with the introduction of generalized
Erdős type spaces. The main theorem here is Theorem 3.4.7. This the-
orem generalizes a result of Dijkstra, stated in Theorem 1.2.5, about
Erdős type subspaces of $\ell^p$, and a result of Dijkstra and van Mill, stated
in Theorem 3.1.2, about Polishable ideals on the natural numbers. In-
deed, at first sight these two subjects are very dissimilar. The space
studied in Theorem 3.4.7 is our so-called generalized Erdős space: it
generalizes the Erdős type spaces in $\ell^p$ of Dijkstra and the Polishable
ideals on the natural numbers studied by Dijkstra and van Mill. Using
a topological characterization of $\mathcal{E}_c$ by Dijkstra and van Mill we use this
theorem to derive some conditions under which this generalized Erdős
space is actually homeomorphic to $\mathcal{E}_c$ in §3.5. In §3.6 we use a topolog-
ical characterization of $\mathcal{E}$ by Dijkstra and van Mill to give conditions
under which our generalized space is homeomorphic to $\mathcal{E}$. Finally, we
prove a fixed point property in §3.7 that generalizes a result of Abry,
Dijkstra and van Mill for the $\ell^p$-case.

In Chapter 4 we delve into the world of nonseparable spaces. This
is motivated by a result of Dijkstra, van Mill and Valkenburg. Whereas
in Chapter 3 we generalize Dijkstra’s theorem about Erdős type sub-
spaces of $\ell^p$ by taking a more general function than the norm function
on a more general (separable metric) product space than the countable
infinite product of the real line, Dijkstra, van Mill and Valkenburg gen-
eralize this result in another way. Extending the norm function to an
uncountable product of the real line, they were able to derive a theorem
similar to that of Dijkstra in this new setting. Using this generalization
and topological characterizations of $\mathcal{E}_c$ by Dijkstra and van Mill they
were able to characterize when their resulting, possibly nonseparable,
Erdős type spaces are homeomorphic to a so-called nonseparable com-
plete Erdős space. Inspired by these results we extend the theorem of
Dijkstra and van Mill about Polishable ideals on the natural numbers, to submeasures on uncountable cardinal numbers. This is the content of Theorem 4.1.2. We are particularly interested in the question when the related ideals are homeomorphic to a nonseparable complete Erdős space. In the last section of Chapter 4 we give a partial answer to this question by showing that for a special class of submeasures the related ideals are homeomorphic to a nonseparable complete Erdős space if and only if the small inductive dimension of these ideals is greater than zero.