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Spectral characterization of the optional quadratic variation process

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Abstract

In this paper we show how the periodogram of a semimartingale can be used to characterize the optional quadratic variation process.

Keywords: semimartingale, quadratic variation, periodogram

1 Introduction and notation

As is well known in the statistical analysis of time series in discrete or continuous time, the periodogram can be used for estimation problems in the frequency domain. It follows from the results of the present paper that the periodogram can also be used to estimate the variance of the innovations of a time series in continuous time. Usually in statistical problems this variance is assumed to be known, since it can be estimated with probability one, given the observations on any nonempty interval in a number of cases. (See for instance Dzhaparidze & Yaglom [4], theorem 2.1).

A fundamental result in another approach is now known as Levy's theorem, which states that the variance of a Brownian Motion can be obtained as the limit of the sum of squares of the increments by taking finer and finer partitions. This result has been generalized by Baxter [1], who showed a similar result for more arbitrary Gaussian processes (that need not to be semimartingales) and to the case where the process under consideration is a semimartingale by Doleans-Dade [2], who obtained a characterization of the quadratic variation. See also theorem VIII.20 of Dellacherie & Meyer [3] or theorem 4 on page 55 of Liptser & Shiryaev [9]. Related work on so called convergence of order p has been conducted by Lepingle [8].
In the present paper we take a different viewpoint towards the quadratic variation process (more in the spirit of theorem 2.1 of [4]) and it is our purpose to show that the periodogram of a semimartingale can be used as a statistic to estimate its quadratic variation process. We thus obtain an alternative characterization of this process as compared to, for instance, Doleans-Dade’s.

The rest of this section is devoted to the introduction of some notation.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be complete filtered probability space and \(X\) real valued semi martingale defined on it. \(X_0\) is assumed to be zero. Let \((\Lambda, \mathcal{L}, Q)\) be an additional probability space. Consider the product \(\Omega \times \Lambda\) and endow it with the product \(\sigma\)-algebra \(\mathcal{F} \otimes \mathcal{L}\) and the product measure \(\mathbb{P} \otimes Q\). Identify \(\mathcal{F}\) with \(\mathcal{F} \otimes \{0, \Lambda\}\) as a \(\sigma\)-algebra on \(\Omega \otimes \Lambda\).

Define for each finite stopping \(T\) and each real number \(\lambda\) the periodogram of \(X\) evaluated at \(T\) by

\[ I_T(X; \lambda) = \left| \int_{[0,T]} e^{i\lambda t} dX_t \right|^2 \]

An application of Ito’s formula gives

\[ I_T(X; \lambda) = 2\Re \left( \int_{[0,T]} \int_{[0.t]} e^{i\lambda(t-s)} dX_s dX_t + [X]_T \right) \tag{1.1} \]

Let \(\xi : \Lambda \to \mathbb{R}\) be a real random variable with an absolutely continuous distribution (w.r.t. Lebesgue measure), that has a density \(G\), which is assumed to be symmetric around zero and consider for any positive real number \(L\) the quantity

\[ E[I_T(X; L\xi)|\mathcal{F}] = E\xi I_T(X; L\xi) := \int_{\mathbb{R}} I_T(X; Lx)G(x)dx \]

It follows from Protter [10], pages 159, 160, that interchanging the integration order in (1.1) is allowed to obtain

\[ E\xi I_T(X; L\xi) = 2 \int_{[0,T]} \int_{[0.t]} g(L(t-s)) dX_s dX_t + [X]_T \tag{1.2} \]

where \(g\) is the (real) characteristic function of \(\xi\).

Our purpose is to study the behaviour of \(E\xi I_T(X; L\xi)\) for \(L \to \infty\). To that end we investigate this quantity for a number of distinguished cases in the next sections.

REMARK: Observe that for all \(s < t\), it holds that \(g(L(t-s)) \to 0\) for \(L \to \infty\), in view of the Riemann-Lebesgue lemma (cf. FELLER II [6]), and (of course) \(|g(x)| \leq 1\) for all real \(x\).

2 Semimartingales of bounded variation

Throughout this section we assume that \(X\) is a process of bounded variation over each finite interval. Denote by \(\|X\|_t\) the variation of the process \(X\) over the interval \([0,t]\) (\(t\) may be replaced by a finite stopping time \(T\)). In this case we obtain from (1.2)
\[ |E_\xi I_T(X; L\xi) - [X]_T| \leq 2 \int_{[0,T]} \int_{[0,t]} |g(L(t-s))|d||X||_t d||X||_t \]  

(2.1)

Since \( \lim_{L \to \infty} g(Lx) = \delta(x) \) (with \( \delta(x) = 1 \) if \( x = 0 \) and \( \delta(x) = 0 \) if \( x \neq 0 \)), an application of Lebesgue's dominated convergence theorem yields that the right hand side of (2.1) converges (almost surely) to

\[ 2 \int_{[0,T]} \int_{[0,t]} \delta(t-s)d||X||_t d||X||_t \]

But this is equal to zero, since \( \delta(t-s) = 0 \) for all \( s < t \), whence the following result:

**PROPOSITION 2.1** Let \( X \) be a semimartingale of bounded variation, \( T \) a finite stopping time and \( \xi \) a real random variable, independent of \( \mathcal{F} \), which has a density on the real line. Then almost surely for \( L \to \infty \)

\[ E_\xi I_T(X; L\xi) \to [X]_T. \]

**REMARK:** Notice the similarity of the above statement with formula 1.5 on page 620 of FELLER II [6], if we take the case where \( \xi \) has a uniform distribution on \([-1, +1]\), and where \( X \) is a piecewise constant process.

### 3 Continuous local martingales

In this section we assume that \( X \) is a continuous local martingale and hence that \( X \) is also locally square integrable. Starting point for our analysis is again equation (1.2). Consider now the process \( Y^X \) defined by

\[ Y^X_t = \int_{[0,t]} \int_{[0,s]} g(L(t-s))dX_s dX_t \]

Observe that \( Y^X \) is again a locally square integrable martingale, and that its predictable variation process is given by

\[ \langle Y^X \rangle_t = \int_{[0,t]} \int_{[0,s]} g(L(t-s))dX_s^2d(X)_t. \]

(3.1)

The first thing we will do in this section is to give an upper bound for the absolute value of the inner integral in (3.1). Thereto we have to introduce some notation. First we need the modulus of continuity of \( X \): \( W_r(\varepsilon) = \sup\{|X_t - X_s| : |t-s| < \varepsilon, s, t \leq r \} \). (\( \tau \) may be a stopping time) Furthermore we have \( X^*_r = \sup\{|X_s| : s \leq t \} \). Next for any function (or process) \( Z \), we denote by \( V(Z; I) \) the total variation of \( Z \) over the interval \( I \). (Notice that \( V(X; I) = \infty \) for any non-empty open \( I \), except for the trivial case when \( X \) is identically zero). We have the following technical result.

**LEMMA 3.1** Let \( X \) be a continuous local martingale, \( g \) a real characteristic function of an absolutely continuous distribution. Then for all \( \varepsilon > 0 \) and \( t > 0 \), the following estimate is valid almost surely
\[ |\int_{[0,t]} g(L(t-s))dX_s| \leq W_t(\varepsilon)(1+V(g;[0,L\varepsilon])) + X_t^*\sup_{t \leq s \leq t}\|g(Ls)| + V(g;[L\varepsilon,Lt]) \tag{3.2} \]

as well as the coarser estimate
\[ |\int_{[0,t]} g(L(t-s))dX_s| \leq 2W_t(\varepsilon)(V(g;[0,\infty)) + 2X_t^*V(g;[L\varepsilon,\infty)) \tag{3.3} \]

**PROOF:** To avoid trivialities, we can assume that both \( V(g;[0,L\varepsilon]) \) and \( V(g;[L\varepsilon,Lt]) \) are finite. Consider first \( \int_{[t-\varepsilon,t]} g(L(t-s))dX_s \). (If \( \varepsilon > t \), then we interpret the integral by extending the definition of \( X \) to the negative real line and setting \( X_t = 0 \) for \( t < 0 \). Integration by parts yields that this integral is equal to
\[
X_t - g(L\varepsilon)X_{t-\varepsilon} - \int_{[t-\varepsilon,t]} (X_s - X_{t-\varepsilon})dg(L(t-s)) - X_t - \int_{[t-\varepsilon,t]} dg(L(t-s)) = X_t - X_{t-\varepsilon} - \int_{[t-\varepsilon,t]} (X_s - X_{t-\varepsilon})dg(L(t-s))
\]

Hence
\[
|\int_{[t-\varepsilon,t]} g(L(t-s))dX_s| \leq |X_t - X_{t-\varepsilon}| + \sup_{t-\varepsilon \leq s \leq t} |X_s - X_{t-\varepsilon}| \int_{[t-\varepsilon,t]} |g||g(L(t-s))|d||L(t-s))
\]
\[ \leq W_t(\varepsilon)(1 + V(g;[0,L\varepsilon])). \]

Consider now the integral over \([0,t-\varepsilon]\). Using again integration by parts, we obtain
\[
|\int_{[0,t-\varepsilon]} g(L(t-s))dX_s| \leq |g(L\varepsilon)X_{t-\varepsilon}| + X_{t-\varepsilon}^*\int_{[0,t-\varepsilon]} d||g||g(L(t-s))|d||L(t-s))
\]
\[ \leq X_{t-\varepsilon}^*|g(L\varepsilon)| + V(g;[L\varepsilon,Lt])). \]

Putting the above two estimates together, we obtain the first statement of the lemma. The second one is a simple consequence, since \( V(g;[0,\infty)) \geq 1, V(g;[0,\infty)) \geq V(g;[0,L\varepsilon]) \) and \( V(g;[L\varepsilon,\infty)) \geq |g(L\varepsilon)| \).

Now it is easy to prove the main result of this section:

**PROPOSITION 3.2** Let \( X \) be a continuous local martingale and let the function \( g \) be of bounded variation over \([0,\infty), \) and \( T \) a finite stopping time. Then

(i) \[ \sup_{t \leq T} \left| \int_{[0,t]} g(L(t-s))dX_s \right| \to 0, \quad \text{a.s.} \]

(ii) \[ E_\xi I_T(X; L\xi) \to [X]_T = \langle X \rangle_T \]

in probability, for \( L \to \infty. \)

\[ \]
PROOF: The first statement easily follows from 3.1 by taking $\varepsilon = L^{-\frac{3}{2}}$, since both $|g(L^{\frac{1}{2}})|$ and $V(g;[L^\frac{1}{2}, \infty)) \to 0$, for $L \to \infty$, as well as $W_T(L^{-\frac{1}{2}}) \to 0$, because $X$ is uniformly continuous on $[0, T]$. In order to prove the second statement, we notice that the following inequality holds:

$$(Y_L)_T \leq (X)_T \sup_{t \leq T} \left| \int_{[0,t]} g(L(t-s))dX_s \right|^2 \to 0 \quad a.s.$$  

Hence a simple application of Lenglart's inequality (cf. Jacod & Shiryaev [7], page 35) yields the result. 

Some examples of distributions for which the conditions of proposition 3.2 are satisfied are the triangular distribution, the double exponential distribution, the Cauchy distribution, the normal distribution (See table 1 of FELLER II [6] on page 503), or the distribution which has the Epanechnikov kernel as its density (This kernel enjoys some optimality properties in problems of kernel density estimation. See e.g. page 21 of [5]). The characteristic function of the uniform distribution on $[-1,+1]$ is not of bounded variation over $[0, \infty)$.

REMARK: It is instructive to see that for deterministic times $T$ in the situation where moreover $X$ has deterministic predictable variation, the proof of the second statement of the above proposition is much simpler and that we don’t need that $g$ is of bounded variation. Indeed consider again $Y^L$ with its quadratic variation given by (3.1). Taking expectations yields

$$E(Y^L)_T = \int_{[0,T]} \int_{[0,t]} g(L(t-s))^2d\langle X \rangle_s d\langle X \rangle_t.$$  

Using again the dominated convergence theorem, we see that $E(Y^L)_T$ tends to zero for $L \to \infty$. So $Y^L_T \to 0$ in probability, in view of Chebychev's inequality.

4 Arbitrary semimartingales

Putting the results of the two previous sections together, we obtain

THEOREM 4.1 Let $X$ be an arbitrary semimartingale, $T$ a finite stopping time and the function $g$ of bounded variation. Then

$$E_{\xi T}(X; L\xi) \to [X]_T$$

in probability, for $L \to \infty$.

PROOF: Let $M$ be the uniquely determined continuous local martingale such that $M_0 = 0$ and such that $Z := X - M$ is of bounded variation over each finite interval. Use the decomposition $X = Z + M$ to split the integral in equation (1.2) into four terms. Apply proposition 3.2 to the term that integrates w.r.t. $dM_t dM_t$, and proposition 2.1 to the term that integrates w.r.t. $dZ_t dZ_t$. Consider then the cross terms. One of them involves the process
\[
\int_{[0,T]} \int_{[0,t]} g(L(t-s))dZ_s dM_t,
\]

which is a local martingale with predictable variation evaluated at \( T \) equal to

\[
\int_{[0,T]} (\int_{[0,t]} g(L(t-s))dZ_s)^2 d\langle M \rangle_t.
\]

This is bounded by

\[
\int_{[0,T]} (\int_{[0,t]} |g(L(t-s))| |d\langle Z \rangle_s|)^2 d\langle M \rangle_t.
\]

Now we use dominated convergence as in the proof of proposition 2.1 to see, that the last double integral tends to zero a.s. As in the proof of proposition 3.2 we finish this part of the proof by again applying Lenglart's inequality.

Next we turn to the other cross term:

\[
\int_{[0,T]} \int_{[0,t]} g(L(t-s))dM_s dZ_t.
\]

whose absolute value can be estimated by above, using lemma 2.1, by

\[
2V(Z; [0,T])(V(g; [0, \infty)) + X_T^* V(g; [L \varepsilon, \infty])).
\]

Take as before \( \varepsilon = L^{-\frac{1}{2}} \) and this expression tends to zero a.s. for \( L \to \infty \). This completes the proof. \( \square \)

As a simple consequence of theorem 4.1 we obtain a representation result for the optional quadratic covaration of two semimartingales.

Let \( X \) and \( Y \) be arbitrary real valued semimartingales, \( T \) a finite stopping time and \( g \) be of bounded variation. Define the cross periodogram of \( X \) and \( Y \) for each real number \( \lambda \) by

\[
I_T(X,Y; \lambda) = \int_{[0,T]} e^{i\lambda t} dX_t \int_{[0,T]} e^{-i\lambda t} dY_t.
\]

Let \( \xi \) be a real random variable as before. Then we have
COROLLARY 4.2 Under the conditions of theorem 4.1 we have

$$E_{\xi} I_T(X, Y; L\xi) \rightarrow [X, Y]_T$$

in probability.

PROOF: It is easy to verify that the following form of the polarization formula holds:

$$I_T(X, Y; \lambda) + I_T(Y, X; \lambda) = \frac{1}{2} [I_T(X + Y; \lambda) - I_T(X - Y; \lambda)].$$

Then an application of theorem 4.1 together with the known polarization formula for the square bracket process and the observation that $E_{\xi} I_T(X, Y; L\xi)$ is real yields the result. \(\square\)

One can define the periodogram for a multivariate semimartingale $X$ with values in $\mathbb{R}^n$ as

$$I_T(X; \lambda) = \int_{[0,T]} e^{i\lambda t} dX_t \int_{[0,T]} e^{i\lambda s} dX_s^*.$$

Then the parallel statement of theorem 4.1 holds in view of corollary 4.2 with $[X]$ the $n \times n$-matrix valued optional quadratic variation process.

We return to real valued semi martingales $X$. Use partial integration to rewrite the periodogram as

$$I_T(X; \lambda) = |e^{i\lambda T} - i\lambda \int_{[0,T]} e^{i\lambda t} dX_t|^2$$

$$= X_T^2 + X_T \int_{[0,T]} i\lambda (e^{i\lambda (T-t)} - e^{-i\lambda (T-t)}) X_t dt + \lambda^2 \int_{[0,T]} e^{i\lambda t} dX_t^2$$

Let $\xi$ be as in the introduction. assume that $E\xi^2 < \infty$. Then $g$ is twice continuously differentiable, so we obtain from the above equation

$$E_{\xi} I_T(X; L\xi) =$$

$$X_T^2 - 2X_T \int_{[0,T]} X_t \frac{\partial}{\partial t} g(L(T - t)) dt + \int_{[0,T]} \int_{[0,T]} X_t X_s \frac{\partial^2}{\partial t \partial s} g(L(t - s)) dt ds \quad (4.1)$$

The idea is that both the two kernels in equation(4.1) behave as a Dirac distribution (although not quite). More precisely we have

PROPOSITION 4.3 Let $X$ be a real semi martingale, $T$ a finite stopping time and $g$ a twice continuously differentiable real characteristic function. The following statements hold almost surely, respectively in probability

$$(i) \int_{[0,T]} X_t \frac{\partial}{\partial t} g(L(T - t)) dt \rightarrow X_T^-$$
\[(ii) \int_{[0,T]} \int_{[0,T]} X_t X_s \frac{\partial^2}{\partial t \partial s} g(L(t - s)) \, dt \, ds \rightarrow X_T^2 + [X]_T.\]

PROOF: (i) follows by partial integration, using the decomposition \(X = M + Z\) as in the proof of theorem 4.1 and an application of proposition 3.2. (ii) is then a consequence of (i) and theorem 4.1.

REMARK: The second statement of this proposition is at first glance perhaps somewhat surprising, since one would expect for continuous \(X\) the term \(X_T^2\) only. The extra term \([X]_T\) is due to the fact, that \(X\) is in general not of bounded variation.

References


