Serie Research Memoranda

A Note On Pareto Laws

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1. ESTEBAN [1986] launches the Weak-Weak Pareto Law (WWPL) for a set \( \mathcal{F} \) of densities \( f \) characterized by, among other things, finite means. He claims that a.) for all \( f \in \mathcal{F} \) this law is weaker than the Weak Pareto Law of MANDELBROT [1960] and b.) the subset \( \mathcal{F}_1 \) of \( \mathcal{F} \) of densities satisfying this law together with two additional conditions, viz. mode existence and a constant declining rate of income elasticity, coincides with \( \mathcal{G} \), the class of generalized gamma densities.

This note shows that:

(I) There exists a class of functions \( \mathcal{Z} \) largely overlapping with \( \mathcal{F} \) in which WWPL is stronger than WPL : \( \mathcal{Z} \cap \mathcal{WWPL} \subset \mathcal{WPL} \).

\( \mathcal{Z} \) includes both functions with and functions without a finite mean.

(II) There are no densities with a constant declining rate of elasticity. As a result, \( \mathcal{F}_1 \) is empty.

2. The Pareto distribution defined as:

\[
F(x) = 1 - (x/x_0)^{-\alpha}, \quad x \geq x_0, \quad \alpha > 0
\]

has lead to the Strong Pareto Law (SPL):

\[
\lim_{x \to \infty} \frac{(x/x_0)^{-\alpha}}{1 - F(x)} = 1, \quad x \geq x_0, \quad \alpha > 0
\]

A weaker version is the Weak Pareto Law (WPL):

\[
\lim_{x \to \infty} \frac{(x/x_0)^{-\alpha}}{1 - F(x)} = 1, \quad x \geq x_0, \quad \alpha > 0
\]

This weaker law, but in a reciprocal form, originates with MANDELBROT [1960]. It is weaker because SPL \( \Rightarrow \) WPL and WPL does not imply SPL as numerous examples may show.
KAKWANI [1980, p.15] defines the elasticity:

\[ K(x) = \frac{x f(x)}{1 - F(x)} \]

As in (2) \( \frac{x}{x_0}^{-\alpha} = \frac{xf(x)}{\alpha} \), an alternative formulation of the laws is:

(5) \( \text{SPL : } \frac{1}{\alpha} K(x) = 1 \quad \text{or} \quad K(x) = \alpha \quad \alpha > 0. \)

(6) \( \text{WPL : } \lim_{x \to \infty} \frac{1}{\alpha} K(x) = 1 \quad \text{or} \quad \lim_{x \to \infty} K(x) = \alpha \quad \alpha > 0. \)

KAKWANI [1980, p.16] writes the derivative of \( K(x) \) as:

(7) \[ K'(x) = \frac{K(x)}{x} \left[ 1 + k(x) + K(x) \right], \]

where \( k(x) = \frac{xf'(x)}{f(x)} \) is called the elasticity of the density by Kakwani.

Elsewhere this term is reserved for \( e(x) = -k(x) \), see ESTEBAN [1978], or for \( \pi(x) = 1 + k(x) \), ESTEBAN [1986], assuming that his definition, see ESTEBAN [1986, p.441] is meant to be:

(8) \[ \frac{1 - \pi(x)}{x} = -\frac{f'(x)}{f(x)} \]

hence with a minus sign on the right hand side.

ESTEBAN [1986] formulates his Weak Weak Pareto Law (WWPL) as

(9) \[ \text{WWPL : } \lim_{x \to \infty} \pi(x) = -\alpha \]

and remarks that both MANDELBROT [1960] and MIRRLEES [1971] take (9) as equivalent to (3).

Esteban himself is of the opinion that (9) is weaker than (3).
3. A link between WPL and WWPL is provided by l'Hospital's Rule, stating (see e.g. RUDIN[1976, p.109]) that if:

A. \( \lim p'(x)/q'(x) \) exists
B. \( q'(x) \neq 0 \)
C. \( \lim p(x) = 0 \)
D. \( \lim q(x) = 0 \)

then \( \lim p(x)/q(x) \) exists and is equal to \( \lim p'(x)/q'(x) \). Substituting

\[
p(x) = xf(x) \quad \Rightarrow \quad p'(x) = f(x) + xf'(x)
\]
\[
q(x) = 1 - F(x) \quad \Rightarrow \quad q'(x) = -f(x)
\]

then gives that if:

- a. \( \lim_{x \to \infty} \frac{f(x) + xf'(x)}{-f(x)} \) exists (WWPL holds)
- b. \( f(x) \neq 0 \)
- c. \( \lim_{x \to \infty} xf(x) = 0 \)
- d. \( \lim_{x \to \infty} [1 - F(x)] = 0 \)

then

\[
(10) \quad \lim_{x \to \infty} K(x) = \lim_{x \to \infty} \frac{x f(x)}{1 - F(x)} = \lim_{x \to \infty} \frac{f(x) + xf'(x)}{-f(x)} = -\lim_{x \to \infty} \pi(x).
\]

Of the four conditions listed above, condition d. is automatically satisfied, while condition b. is one of the characteristics assumed by Esteban for his densities.

Hence, if \( Z \) is defined as \( Z = \{ f | \lim_{x \to \infty} xf(x) = 0 \} \), then for all \( f \in Z \) satisfying WWPL (condition a.) according to (10) WPL also applies. In other words, for all \( f \in Z \) we have WWPL \( \Rightarrow \) WPL. As there exists at least one \( f \in Z \) for which WPL applies but WWPL does not (see Lemma 1 in the Appendix), the reverse implication does not hold. This proves the first part of statement I.

For the second part, we note that the class \( Z \) includes Pareto densities with \( 0 < \alpha \leq 1 \), which do not have a finite mean, and for which both WWPL and WPL hold. Conversely, there are many functions with a finite mean which are also in \( Z \). This proves the second part of statement I.
4. ESTEBAN [1986] confines his analysis to \( \mathcal{F} \), defined as the class of density functions \( f \) with support \([a, b] \), \( 0 \leq a < b = \infty \), such that: i) \( f \) has finite mean, ii) \( f \) is \( C^1 \) in \((a, b)\), and iii) \( f(x) > 0 \) for all \( x \in (a, b) \), and \( f(x) = 0 \) otherwise.

We have not used this class for several reasons. In the first place, the requirement that the density has a finite mean is not used explicitly by Esteban in his analysis, and is not a natural property to require of a continuous income distribution function. Second, we note with interest that if a function satisfies WPL it must be a member of our class \( Z \). This can be seen by noting that if \( \lim_{x \to \infty} x f(x)/[1 - F(x)] \) is finite, then \( \lim_{x \to \infty} x f(x) \) must be 0. In other words, WPL \( \subset \mathcal{Z} \) and \( \mathcal{Z} \) could thus be considered a better candidate for a "Weak Weak Pareto Law" than Esteban's WWPL, for which a similar property does not hold, see Lemma 1 in the Appendix.

We note in passing that \( \mathcal{Z} \) is not a subset of \( \mathcal{F} \) (as there are functions with a finite mean for which \( x f(x) \) does not approach 0, see Lemma 2 in the Appendix) and that \( \mathcal{F} \) is not a subset of \( \mathcal{Z} \) (as e.g. the aforementioned Pareto densities with \( 0 < \alpha \leq 1 \) show). We still doubt if there is any density which satisfies WWPL but which does not satisfy WPL. The above result shows that such a density must be sought outside \( \mathcal{Z} \), and we might conjecture that no such density exists.

5. There are many densities in \( \mathcal{F} \) that do not satisfy either WPL or WWPL. A spicy example is the two parameter Weibull (or Extreme-value) distribution,

\[
F(x) = 1 - e^{-\gamma x^p} \quad x \geq 0, \quad p > 0, \quad \gamma > 0
\]

the simplest member of the family discovered by WEIBULL [1951]. Its density is

\[
f(x) = p \gamma x^{p-1} e^{-\gamma x^p}.
\]

As the mean \( \mu = \left( \frac{1}{p} \right)^{1/p} \Gamma(1/p + 1) \) is finite within the given domain of \( \gamma \) and \( p \), this density is clearly in \( \mathcal{F} \). For this density we have from (4):

\[
K(x) = \gamma px^p,
\]

which for \( x \to \infty \) approaches infinity for \( p > 0 \). Hence WPL does not apply.
Acknowledging that

\[ (14) \quad \pi(x) = \frac{xK'(x)}{K(x)} - K(x) = p - K(x) \]

we get:

\[ (15) \quad \pi(x) = p - \gamma px^p, \]

which for \( x \to \infty \) and \( p > 0 \) does not converge to a finite value. Hence for the two parameter Weibull with \( p > 0 \) neither WPL nor WWPL applies.

The Weibull distribution has a mirror image in the Inverse Weibull:

\[ (16) \quad F(x) = e^{-\gamma x^p} \quad x \geq 0, \quad p < 0, \quad \gamma > 0 \]

\[ (17) \quad f(x) = -p\gamma x^{p-1}e^{-\gamma x^p} \]

Using (4) we get:

\[ (18) \quad K(x) = -p\gamma x^{p-1}e^{-\gamma x^p}\left[1 - e^{-\gamma x^p}\right] \xrightarrow{\text{I'Hospital}} -p \quad \text{as} \quad x \to \infty \]

while (8) gives:

\[ (19) \quad \pi(x) = p - \gamma px^p \xrightarrow{\text{as} \quad x \to \infty} p \]

The Inverse Weibull therefore satisfies both WWPL and WPL. Note that we could have confined ourselves to proving WWPL, as the Inverse Weibull is in \( Z \).

The two parameter Weibull, considered above, is a special case (\( q=1 \)) of the generalized gamma density, discovered by STACY [1962]:

\[ (20) \quad f(x) = |p|\gamma x^{pq-1}e^{-\gamma x^p/\Gamma(q)} \quad x \geq 0, \quad p \neq 0, \quad q > 0, \quad \gamma > 0 \]

If Esteban's statement were true the class \( \mathcal{F} \) would be characterised by (20). These densities would satisfy WWPL and give rise to a declining elasticity function \( \pi(x) \), whatever the value of \( q \) and \( p \). But, as the example above shows, this is clearly not true for values of \( p > 0 \) and \( q = 1 \).

This gives an indication that Esteban's statement that \( \mathcal{F}_1 \) coincides with \( \mathcal{F} \) is incorrect.
6. ESTEBAN [1986] searches for distributions satisfying the following three hypotheses:

Hypothesis 1: WWPL
Hypothesis 2: The mode m exists.
Hypothesis 3: \( \pi(x) \) has a constant declining rate.

Distributions satisfy hypothesis 3, if

\[
\frac{d \ln \pi'(x)}{d \ln x} = \delta \quad \delta \leq 0.
\]

or

\[
d \ln \pi'(x) = \delta \, d \ln x \quad \delta \leq 0.
\]

Integration gives:

\[
\ln \pi'(x) = \delta \ln x + \ln \mu \quad \mu > 0, \quad \delta \leq 0
\]

\[
\pi'(x) = \mu x^\delta
\]

\[
d \pi(x) = \mu x^\delta \, dx \quad \mu > 0, \quad \delta \leq 0
\]

Further integration gives

\[
\pi(x) = -\alpha + \frac{\mu}{\delta + 1} x^{\delta + 1}
\]

if \( \delta \leq 0 \) but \( \neq -1 \), \( \alpha \in \mathbb{R} \), \( \mu > 0 \)

\[
\pi(x) = -\alpha + \mu \ln x
\]

if \( \delta = -1 \), \( \alpha \in \mathbb{R} \), \( \mu > 0 \).

(and not \( \pi(x) = -\alpha + \mu / \ln x \), see ESTEBAN [1986 formula (5) on p.443])

As

\[
\pi(x) = 1 + \frac{d \ln f(x)}{d \ln x}
\]

the concomitant densities are obtained by integration. For (24) this gives:

\[
d \ln f(x) = (-1 - \alpha + \mu \ln x) \, d \ln x
\]

or

\[
f(x) = \lambda x^{-1-\alpha} \exp\left(\frac{1}{\mu} \ln^2 x\right) \quad \mu > 0, \ \lambda > 0
\]

which cannot be a density as \( \lim_{x \to \infty} f(x) = 0 \).
For (23) we have:

\[ d \ln f(x) = (-1 - \alpha + \frac{\mu}{\delta+1} x^{\delta+1}) d \ln x \]

\[ = (-1 - \alpha) d \ln x + \frac{\mu}{\delta+1} x^{\delta+1} d \ln x \]

and thus:

\[ f(x) = \lambda x^{-1-\alpha} \exp\left(\frac{\mu}{\delta+1} x^{\delta+1}\right) \]

\[ \delta \leq 0, \; \delta \neq -1, \; \mu > 0, \; \lambda > 0 \]

(Note that \( \alpha \in \mathbb{R} \) and not \( \alpha > 0 \) and \( \epsilon = -(\delta + 1) > -1 \) and not \( \epsilon > 0 \), see ESTEBAN [1986, p.444])

Comparison with (20) shows:

\[ \delta + 1 \quad = \quad p \quad \delta \leq 0, \; \delta \neq -1 \quad \Rightarrow \quad p \leq 1, \; p \neq 0 \]

\[ \mu \quad = \quad -\gamma \rho^2 \quad \mu > 0 \quad \Rightarrow \quad \gamma < 0 \]

\[ \alpha \quad = \quad -\rho q \quad \alpha \in \mathbb{R} \quad \Rightarrow \quad q \in \mathbb{R} \]

\[ \lambda \quad = \quad |p| \gamma^q / \Gamma(q) \quad \lambda > 0 \quad \Rightarrow \quad q \in \mathbb{N} \]

However, for \( \gamma < 0 \) (20) does not represent a density. Hence there is no density with a constant declining rate of elasticity. The conclusion is that the set \( \mathcal{F}_1 \) of densities satisfying all of ESTEBAN's requirements is empty.

This proves statement II.

7. Since Esteban's third demand is impossible to satisfy, we can drop it. We are then left with two demands for densities to satisfy: WWPL and the existence of a mode. Obviously there are many densities satisfying these two demands, the Inverse Weibull being an example. In fact, all generalised gamma densities with \( p < 0 \) fall into this category. These functions are usually called Inverse Gamma Functions, or Pearson's Type V, see RAIFFA & SCHLAIFER [1961, chapter 7]. These densities also lie in \( \mathcal{F} \) as their mean is \( \frac{1}{\gamma} \Gamma(\frac{p+1}{p}) / \Gamma(q) \) which is finite in the relevant domain, and also in \( \mathbb{Z} \).

The validity of WPL for the set of inverse gamma functions had already been established by KLOEK and VAN DIJK [1976], see also KLOEK and VAN DIJK [1978].

8. Finally it must be added that WPL is an observed phenomenon and not a requirement from theory, see MERKIES [1987]. This means that if some density is a good approximation of an empirical frequency distribution, it is irrelevant whether it also satisfies WPL (or WWPL for that matter).
Appendix : Two counterexamples

NOTE : unless otherwise stated, all densities considered here are defined as follows : \( f(x) > 0 \ \forall \ x \in [0,\infty), \) \( f(x) = 0 \ \forall \ x < 0. \) \( f \in C^1. \)

We will find the Weibull-representation of densities (see WEIBULL [1951]) useful :

\[
F(x) = 1 - \exp[-\phi(x)]
\]

with \( \phi(x) \) satisfying the following four conditions :

1. \( \phi(x) > 0 \)
2. \( \phi(x) \) nondescending
3. \( \phi(0) = 0 \)
4. \( \phi(\infty) = \infty \)

A sufficient condition for \( f \in C^1 \) is that \( \phi(x) \in C^2. \) Combined with condition 2. above we then get as corollary that \( \phi'(x) \geq 0. \)

We find that

\[
f(x) = \phi'(x)\exp[-\phi(x)]
\]

\[
K(x) = xf(x)[1-F(x)]^{-1} = x\phi'(x)
\]

\[
\pi(x) = 1 + x\phi''(x)/\phi(x) = 1 - x\phi'(x) + x\phi''(x)/\phi'(x)
\]

We can now reformulate in terms of this representation :

Weak Pareto Law : (Mandelbrot) \hspace{1cm} (notation : \( f \in W_1 \))

\[
K(x) \to \alpha > 0 \quad \Rightarrow \\
\pi(x) \to \alpha \quad \Rightarrow \\
\phi'(x) \to 0
\]

Weak Weak Pareto Law : (Esteban) \hspace{1cm} (notation : \( f \in W_2 \))

\[
\pi(x) \to \beta < 0 \quad \Rightarrow \\
1 - x\phi'(x) + x\phi''(x)/\phi'(x) \to \beta
\]

\( \mathcal{F} \)-class : (Esteban) \hspace{1cm} (notation : \( f \in \mathcal{F} \))

\( f(x) \) has a finite mean

\( Z \)-class : (Merkies & Steyn) \hspace{1cm} (notation : \( f \in Z \))

\[
xf(x) \to 0 \quad \Rightarrow \\
x\phi'(x)\exp[-\phi(x)] \to 0
\]
Lemma 1: $W_1 \setminus W_2 = \emptyset$

Proof

We can rephrase this Lemma as $W_1 \setminus (W_2 \cap W_1) = \emptyset$. Note that for $f \in W_2 \cap W_1$,

$1 - xp'(x) + xp''(x)/p'(x) \to \beta$ and $xp'(x) \to \alpha$. This leads to the following useful result:

$f \in (W_1 \cap W_2) \Rightarrow x^2 p''(x) \to \alpha(\alpha + \beta - 1) \Rightarrow \gamma$

We will therefore construct a density $f$ with the following properties:

$(xp'(x) \to \alpha > 0) \land \gamma(x^2 p''(x) \to \gamma)$

This is, of course, in addition to the four properties for the Weibull representation listed at the start of the Appendix.

A function $p'(x)$ satisfying these requirements is

$p'(x) = \alpha x^{-1} + (1 + \sin x)x^{-2}$

with

$p''(x) = -\alpha x^{-2} + x^{-2} \cos x - 2(1 + \sin x)x^{-3}$

since

$xp'(x) = \alpha + (1 + \sin x)x^{-1} \to \alpha$

$x^2 p''(x) = -\alpha + \cos x - 2(1 + \sin x)x^{-1} \to -\alpha + \cos x$

We therefore have the required counterexample by taking $f = p'(x)exp[-p(x)]$, with $p'(x)$ as above. $f$ can not be written explicitly, and we must still check the other demands placed on the function $p(x)$. Note that $p'(x) > 0 \\forall x$, so $p(x)$ is nondecreasing. To prove that $p(\omega) = \infty$, we note that:

$\alpha \ln x \leq \alpha \ln x + \int (1 + \sin x)x^{-2}dx$

or

$\alpha \ln x \leq p(x)$

proving that $p(x)$ approaches infinity as $x$ increases.

The exact behaviour of $p(x)$ near 0 is not crucial. Since it is a nondecreasing function there are three possibilities:

1. $p(x) \to 0$. In which case all is well without further ado.
2. $p(x) \to C \neq 0$, in which case we use $p^*(x) = p(x) - C$ instead
3. $p(x) \to -\infty$. In that case, there is a $x_0 > 0$ with $p(x_0) = 0$. We therefore use $p^*(x) = p(x-x_0)$ instead.

Through this construction, the condition that $p(x) > 0$ is also satisfied.

Finally, we note that $p(x) \in C^2$ and that the associated density $f(x) > 0 \\forall x$.

The end result is a function $p(x)$ and an associated density $f(x)$ which satisfies all our requirements.

\[\blacksquare\]
Lemma 2: $\mathcal{F} \setminus \mathbb{Z} \neq \emptyset$

Proof
We construct a density with a finite mean for which $xf(x)$ does not approach zero. Initially we will construct this density on $[1, \infty)$, a density on $[0, \infty)$ then follows easily.

We will use the following functions as building blocks:

$$b(x; a, n) = (2\pi)^{-1} \left( 1 - \cos \left( \frac{x-n}{a} \right) \right) I(x, [n, n+2\pi a])$$

where $I(x, A) = 1$ if $x \in A$, and $0$ otherwise.

This function has the following properties:

$$b(x; a, n) \geq 0 \quad \int b(x; a, n) \, dx = a \max b(x; a, n) = 1/\pi$$

As first intermediate result we construct a function

$$h(x) = \sum_{i=1}^{\infty} b(x; a_i, n_i)$$

with $\sum a_i = S < \infty$ and $\lim_{n \to \infty} n_i = \infty$

To avoid the individual building blocks overlapping we also require that

$$n_i \geq n_{i+1} + 2\pi a_i$$

and $n_1 = 1$

Note that $\int h(x) \, dx = S$.

As our next intermediate we take

$$g(x) = S^{-1} h(x)$$

which is a density function. The density function we seek for our counterexample then follows as:

$$f(x) = Ag(x)/x$$

with $A^{-1} = \int g(u)/u \, du$

The latter integral exists, as $0 < g(u)/u < g(u)$ for $u>1$, and $\int g(u) \, du$ exists.

We note that:

$$xf(x) = Ag(x) = AS^{-1} h(x)$$

This function continues to vary between $0$ and $AS^{-1} \pi^{-1}$ and therefore does not approach $0$. Therefore, $f \notin \mathbb{Z}$. However,

$$Ex = \int xf(x) \, dx = AS^{-1} \int h(x) \, dx = A < \infty$$

Therefore $f \in \mathcal{F}$. Hence $\mathcal{F} \setminus \mathbb{Z} \neq \emptyset$. \qed
References:


Raiffa, H. and R. Schlaifer, "Applied statistical decision theory", (1961), Graduate School of Business Administration, Harvard University, Cambridge MA.


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