ON THE EQUITABLE DISTRIBUTION OF THE HOUSING STOCK

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ON THE EQUITABLE DISTRIBUTION OF THE HOUSING STOCK

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Abstract

In this paper we study the relation between the distribution of income and the distribution of a given housing stock over under various allocation mechanisms. In order to study a pure situation the population consists of households which differ only in income. We analyze the effects of various government measures intended to change the relation between these distributions.

We first study a system in which prices are the only allocation instrument and switch later on to more complicated allocation mechanisms in which part of the costs associated with housing are financed by tax payments and a rent subsidy is introduced. Finally we consider a system of income prices.

The positive relationship between income and housing quality can only be altered when the rents are dependent on income because of a rent subsidy or income prices. However, even under such allocation mechanisms the positive relationship holds when housing is a luxury good.

In order to judge the effects of the various allocation mechanisms we employ the theory of fairness as developed by Kolm, Varian and others. It is shown that the distribution of the housing stock under uniform prices, i.e. prices which are independent of income, is the only one that can be considered as incrementally fair. Allocation systems which use prices that are dependent on income do not have this property.
1 Introduction

In many West European countries the government has taken a number of measures that influence the functioning of the housing market. The immediate purpose of this intervention is usually the improvement of market outcomes, in the sense that one tries to reach a more equitable distribution of housing costs and housing possibilities over the population of households. Although the number and intensity of policy measures referring to the housing market varies from country to country (government intervention seems to be most intense in Sweden and the Netherlands), their influence on the allocation of available dwellings over the population is everywhere regarded as significant.

The main aspect of the perceived inequity concerns the discrepancy between perceived housing needs and housing possibilities. These housing needs should not be identified with housing demand in the sense that is given to this term in economics. 'Housing needs' usually refers to a concept of reasonable minimal requirements which the housing situation of households of a particular composition should fulfil. The concept is hard to define in a precise way and its actual content seems to vary with the welfare level of the society concerned as well as with the situation of the government budget. Notwithstanding the vagueness of the concept, the perceived housing needs play an important role in the determination of the housing policy in West European countries.¹

One of the main reasons for making housing needs, as distinguished from housing demand, an object for government concern seems to be the fact that demand is influenced by the income of the household concerned as well as by its preferences. The common-sense reasoning seems to be that households with the same composition but different incomes have equal needs for housing, but differ in the possibilities they have to realize this need. It is indeed difficult to deny that such reasoning has some appeal.

Much of the political concern for the housing market has to do with the relation between a household's income and the dwelling it occupies, and with the possibilities to influence this relation. For this reason we will in this paper study the relation between the distribution of income and the allocation of the housing stock over the population under various allocation mechanisms. These allocation mechanisms all respect the freedom
of choice of individual households, but provide different incentives for actual behaviour. The question we seek to answer is whether significant changes in the relation between income and housing can be the result of such indirect ‘guidance’ of the housing market.

We will use a highly stylized model in which a population of households which differ only in the income they earn has to be distributed over a given, heterogeneous housing stock. Freedom of choice is respected and in equilibrium demand should be equal to supply for each available dwelling type. It will also be assumed that the government budget, as far as it concerns housing policy, is always balanced. It will be clear from this short description that the model cannot be viewed as a close representation of a real-world housing market. Instead, it is intended to give a clear representation of a pure case in which our problem can be analyzed. In the final section of the paper the relevance of this analysis will be discussed and extensions suggested.

In the next section we will - as a prologue - shortly discuss the preoccupation of many politicians with the relation between income and housing from the economist’s point of view. We then move on to a formal presentation of our model and discuss household behaviour, market demand and equilibrium. Then four different allocation mechanisms will be introduced and compared with respect to their effects on the distribution of the housing stock and the distribution of income-net-of-housing-costs. The paper is concluded with some remarks on the normative aspects of these mechanisms.

2 Housing Policy and Housing Needs

The purpose of social policy, including housing policy, is usually the reduction of social inequality. Social inequality has a number of aspects, of which inequality on the housing market is only one, but it is well-known that there exist some general relationships between apparently different indicators of this phenomenon. For instance, well-educated people in general receive a relatively high income and live in a comfortable dwelling. The correlation between the relative position of households as measured by various indicators implies that society functions more or less as a filter with overall losers and winners. This can be interpreted in the following way: men, which are to viewed as equal in some fundamental sense, are allowed to become unequal by the way society functions. Although
the normative judgments that this process invokes differ, it is no exaggeration to say that many people regard this as unjust. As a consequence there is broad support for government measures that are intended to reduce the resulting inequality to some extent. This aspiration is especially present among social-democrats, but - although usually to a smaller extent - by other political denominations as well.

The particular aspect of social inequality that can be observed on the housing market has been described by Priemus, an authoritative investigator of the Dutch housing market and also a member of the Dutch labour party, as the widely observable

'phenomenon that strong participants (in the housing market) are in general able to occupy a dwelling which is placed high in the hierarchy of dwelling types, while weaker participants have to be satisfied with dwelling types which are placed low in this hierarchy' [Priemus, 1983, p.278].

Although the strength of a participant's position on the housing market mentioned here is certainly not identical with its income position, the interrelatedness of the various aspects of social inequality makes it, nevertheless, a reasonable indicator. The relationship can therefore be formulated in pregnant form (see Priemus [1978], [1983]) as the 'iron law of the housing market':

'occupiers with the highest incomes live in the best dwellings, those with the lowest incomes in the worst dwellings' [Priemus, 1983, p.278].

This 'iron law' will be of major concern in the remainder of this paper. Government measures relating to the housing market are usually intended to mitigate this relation. The effects of such measures may be two-fold: the distribution of income-net-of-housing-costs may become more equal (as compared to market allocation) as a result of the special measures taken to help the poor and the distribution of the dwelling types over the households may become more equitable. Moreover, the interrelatedness of the various aspects of social inequality gives some hope that additional positive effects may occur as well.

For an economist the empirical observation of the 'iron law' may invoke the conclusion that housing is clearly a non-inferior good (as is confirmed by all available empirical evidence) and that - for this reason -
(relatively) rich households consume more of it than (relatively) poor ones. This does not in itself provide any more reason for government intervention than does the fact that there is e.g., a positive correlation between income and holiday expenditures. If one does not like the inequality in the allocation on the housing market one should change the cause of this particular allocation, viz. the unequal income distribution. Measures that refer to the housing market in particular (e.g. rent subsidies) should on the basis of conventional micro-economic theory be expected to be less efficient than measures that change the income distribution directly by means of lump-sum transfers. Many economists would therefore be inclined to advice the government to introduce (a more severe form of) income policy and to abstain from housing policy.

Although this opinion is attractive, and indeed irrefutable, from the purely theoretical point of view which presumes a competitive world with Pareto-optimal outcomes and an initial distribution of resources that can be changed ad valorem, its practical relevance may be doubted.\(^3\) The reduction of actual income inequality is not an easy task and will probably have (large) negative side-effects. In the actual world of second best choices it may therefore be a good policy on the one hand to maintain some income inequality, while on the other to mitigate some of its (perceived) negative consequences. One of these ill-preferred effects is that for households with the lowest incomes the sufficient provision of some basic needs, such as housing, is not guaranteed by the market. The normative question, then, is whether social welfare can be increased by government interference in the housing market.

The importance of this question should not be obscured by the fact that we do not have an unambiguous and operational concept of social welfare. The Pareto-criterion is obviously unsatisfactory in questions of distributive justice. Utilitarianism also has its well-known drawbacks. The best alternative seems to be the concept of fairness, as developed by Kolm [1972], Varian [1974] and others. It considers the distribution of a set of commodities over a population as 'fair' if none of the participants envies the share any other participant gets (i.e. if he regards his own share as at least as good as any other). We will use this theory for the comparison of the effects of the various allocation mechanisms.

An alternative way of looking at the effects of the various possible measures influencing the housing market takes the viewpoint that
politicians are concerned with the number of votes. The various possible measures that can be taken are probably all beneficial to some and disadvantageous to others. The attractiveness of possible measures is therefore dependent on the distribution of the costs and benefits associated with it over the population. This point of view might facilitate the study of actual policy measures taken in various countries.

It may be concluded from the foregoing that there is reason - both from the normative and the positive point of view - to study the relation between income and housing. In this paper we will try to contribute to such analysis. We will start in the next section with a formal presentation of our frame of analysis. Subsequent sections are devoted to an analysis of the relation between the various possible allocation mechanisms and the resulting allocation on the housing market. In the final section we return to the normative aspects of the problem under consideration.

3 The Allocation Problem

Throughout this paper we will be concerned with the question of how to distribute a given stock of heterogeneous dwellings over a given population of b households, which differ only in income. Income will be regarded as a continuous variable and the income distribution will be described by a continuous density function f(y) which is assumed to have positive support on some interval \([y_{\min}, y_{\max}], y_{\max} - y_{\min} > 0\), and is zero elsewhere. The set of households earning an income \(y\) is always assumed to be of measure 0. It will be assumed that there is a finite number, \(N\), of dwelling types and we will refer to the total number of dwellings of type \(n\) as \(S_n\). These numbers are exogenously determined and we will assume the following:

Assumption 1

\[
\sum_{n=1}^{N} S_n = b. \tag{1}
\]

This assumption says that the market is balanced, i.e. total demand is exactly equal to total supply. In the present context, where we are not concerned with dynamics and mobility, it is hard to see why there should be more dwellings than households to occupy them.

We use the general assumption that prices are determined by allocation
variables \( \omega \) and by the income of the household:

\[
p_n = p_n(\omega_n, y), \quad n=1, \ldots, N. \tag{2}
\]

The function \( p_n \) is increasing in \( \omega_n \) for all income levels. It should be interpreted as the price a household with income \( y \) has to pay for a dwelling of type \( n \). This implies that it is no longer meaningful to speak of the price of a particular type of dwelling. A price equilibrium is therefore no longer characterized by a set of values, but by a set of functions, viz. the \( p_n \)'s. Since it is, nevertheless, convenient to have the possibility to characterize a price equilibrium by a set of values, we have introduced the variables \( \omega_n \), which will be referred to as allocation variables. In later sections of the paper particular interpretations of the allocation variables will be given.

We will assume throughout that, for each income \( y \), \( p_n \) becomes infinitely large if \( \omega_n \) increases without an upper bound and becomes infinitely small (negative) if \( \omega_n \) decreases without a lower bound.

Sometimes it will be convenient to regard the price \( p_n \) as consisting of two parts: the rent, to be denoted as \( r_n \), and taxes or subsidies associated with housing, to be denoted as \( q \) (note that the latter variable is independent of the dwelling type \( n \)). Only the rental part of the price \( p_n \) depends on \( \omega_n \), and we have:

\[
p_n(\omega_n, y) = r_n(\omega_n, y) + q(y), \quad n=1, \ldots, N. \tag{3}
\]

The determination of the tax \( q(y) \) will be discussed later on. Note that we do not restrict the rents or taxes to be nonnegative. When they are negative, they can be interpreted as monetary compensations for a relatively low dwelling quality or as subsidies. Examples will be encountered in later sections.

Households are assumed to be utility maximizers and choose the dwelling type which offers them the highest level of utility. They will always choose one type of dwelling (not two or more, or zero). The total number of households that choose a dwelling of type \( n \) will be referred to as the demand \( D_n \). A first condition which any equilibrium allocation to be considered here has to fulfil is that it should be market-like, i.e. demand should be equal to supply for all types of dwellings:

\[
D_n = \sum_{n=1}^{N} D_n(\omega_n, y) = \sum_{n=1}^{N} r_n(\omega_n, y) + q(y),
\]

\[
x_n = \frac{1}{\sum_{n=1}^{N} D_n(\omega_n, y)} \sum_{n=1}^{N} D_n(\omega_n, y),
\]

\[
\Rightarrow \sum_{n=1}^{N} x_n = 1.
\]
The stock of dwellings is assumed to be completely owned by the government, which also sets the rules for its distribution over the households. The costs of maintenance of the housing stock have to be paid by the government and an essential second condition which any allocation mechanism to be considered here should satisfy is that the revenues of the housing stock are sufficient to pay these costs. The cost of maintenance of a dwelling of type \( n \) will be denoted as \( c_n \). They do not need to be equal to the rent \( r_n \) that has to be paid for this dwelling. What is always required, however, is that the sum of the revenues equals the sum of the costs. This implies:

\[
\sum_{n=1}^{N} \int p_n(\omega, y) \delta_n(\omega, y) f(y) b(y) dy = \sum_{n=1}^{N} c_n S_n, \tag{5}
\]

where \( \delta_n \) is a dummy variable which takes on the value 1 if a household demands a dwelling of type \( n \) and is equal to zero otherwise, and \( \omega \) is the vector of allocation variables. The determination of the \( \delta_n \)'s will be discussed in the next section.

Equation (5) states that the costs of maintenance of the housing stock are completely paid in each period. The budget of the government is therefore always assumed to be balanced, as far as it concerns housing expenditures. This is our second equilibrium condition.

The foregoing discussion motivates the following:

**Definition** An equilibrium is a vector of allocation variables \( \omega \) such that:

a) demand equals supply for all dwelling types 1...N,

b) the government budget is balanced.

For later reference we introduce the following notational conventions: the left-hand-side (lhs) of equation (5) will sometimes be denoted briefly as \( P \):
\[ P = \sum_{n=1}^{N} \int p_n(\omega_n, y) \delta_n(\omega, y) f(y) b \, dy, \]  
(6)

The right-hand-side (rhs) as \( C \):

\[ C = \sum_{n=1}^{N} c_n S_n. \]  
(7)

The revenues \( P \) consist of two parts, as becomes clear by substitution of (3) into (6):

\[ P = \sum_{n=1}^{N} \int r_n(y, \omega_n) \delta_n(y, \omega) f(y) b \, dy + \int q(y) f(y) b \, dy. \]  
(8)

The first term on the rhs of this equation will sometimes be denoted as \( R \):

\[ R = \sum_{n=1}^{N} \int r_n(y, \omega_n) \delta_n(y, \omega) f(y) b \, dy, \]  
(9)

the second term as \( Q \):

\[ Q = \int q(y) f(y) b \, dy. \]  
(10)

4 Household Behaviour and the Existence of Equilibrium

It has already been mentioned in the preceding section that we will consider a population of utility-maximizing households which is homogeneous except for one characteristic: income. Since this is the purest situation one can imagine in which our problem occurs it is a convenient starting point for the present analysis.

The utility that is experienced by a household is a function of the vector characteristics \( d \) of the dwelling it occupies and of the vector \( x \) of the quantities of other commodities which it consumes:

\[ u = u(d, x). \]  
(11)
We will distinguish a finite number, \( N \), of dwelling types and the vector \( d \) is therefore restricted to take on only \( N \) values, each corresponding with the characteristics of one available dwelling type.

In previous sections we have spoken of best and worst dwellings, without making explicit how the appropriate ranking of the dwelling types could be determined. The formulation of the utility function in (11) does not assure that all households rank all dwellings in the same way. This is caused by the fact that the availability of other consumption goods may influence the ranking of the dwelling types. For instance, it may happen that a household that possesses a car has other preferences about the location of the dwelling it occupies than a household that does not.

In order to rule out this possibility and be able to define unambiguously a hierarchy of dwelling types, it is necessary to assume that the preferences of the households are weakly separable in the dwelling characteristics, i.e. that the utility function can be written as:

\[
\begin{align*}
\text{u}(d, x) &= \text{u}'(w(d), x).
\end{align*}
\]  

(12)

The function \( u \) should be increasing in \( w(d) \). The latter function can be interpreted as a partial utility function for dwelling characteristics.

We summarize this as:

**Assumption 2** The preferences of the households are weakly separable in the dwelling characteristics.

Assumption 2 allows us to rank the dwellings on the basis of their partial utilities. Dwellings of type 1 are the most desirable, dwellings of type \( N \) least desirable.

For simplicity of notation we will use \( w_n \) as an abbreviation of \( w(d_n) \) throughout the remainder of the paper.

In order to determine the household's demand for housing the utility function has to be maximized subject to the budget constraint. This constraint can be written as:
\[
\sum_{n=1}^{N} \delta_n p_n(\omega_n y) + \pi x \leq y, \quad n=1, \ldots, N,
\]

where the \( \delta_n \)'s are dummy variables which take on the value 1 when a dwelling of type \( n \) is chosen and are equal to zero otherwise, and \( \pi \) denotes the prices of the other commodities. It will be assumed throughout that preferences are such that only one type of dwelling will be chosen.4

On the basis of the foregoing we cannot be sure that the will be dwellings for which the budget constraint is satisfied. For instance, one can imagine the situation in which all prices \( p_n \) exceed \( y \) for some income levels. It is however possible to exclude such situations when we make the following assumption:

**Assumption 3**

\[
\sum_{n=1}^{N} c_n s_n < y_{\text{min}}, \quad (14)
\]

This assumption says that the average costs of housing do not exceed the minimum income level \( y_{\text{min}} \). The budget constraint of the government guarantees that there must be at least one dwelling type for which \( p_n < y_{\text{min}} \), which is sufficient to ensure that every household can choose a dwelling type without violating his budget constraint.

The maximum level of utility that can be obtained when \( \delta_n \) is restricted to take on the value 1 will be denoted as \( v_n \) and is a function of the partial utility of housing, of income minus the rent of the dwelling and of the prices of other consumption goods:

\[
v_n = v(w_n, y-p_n, \pi), \quad n=1 \ldots N.
\]

The variable \( v_n \) is the indirect utility associated with the occupation of a dwelling of type \( n \). Since the prices \( \pi \) are the same for all households and we are, in the present paper, not concerned with the consequences of changes in them, they will usually be suppressed. We make the following assumption5:

**Assumption 4** The functions \( v_n \) are continuously increasing in \( w_n \) and \( y-p_n \).
It follows from this assumption that it is possible to substitute (small) changes in income for (small) changes in housing quality and vice versa. In what follows, we will need a stronger, global, notion of this substitutability. For this reason we formulate:

**Assumption 5** For all possible values of $y$, $w_i$, $w_j$ and $p_i$ there exists a $p_j^*$ such that:

$$\begin{align*}
\nu(w_i, y-p_i) &< \nu(w_j, y-p_j^*) \text{ whenever } p_j^* < p_j^* \\
\nu(w_i, y-p_i) &= \nu(w_j, y-p_j^*) \\
\nu(w_i, y-p_i) &> \nu(w_j, y-p_j^*) \text{ whenever } p_j^* > p_j^* 
\end{align*}$$

This assumption states that differences between the qualities of two dwelling types can always be compensated by differences in income-net-of-housing-costs. Note that we do not exclude negative prices.

The dwelling type that will be demanded by the household is the one that gives the highest level of utility $\nu_n$:

$$\delta_n = \begin{cases} 
1 & \text{if } \nu_n \geq \nu_{n'}, \text{ for all } n' \in \{1 \ldots N\} \\
0 & \text{otherwise}
\end{cases} \quad (17)$$

This implies that we will have $\sum_{n=1}^{N} \delta_n = 1$, except for those cases in which the maximum value of $\nu_n$ is reached for two or more dwelling types. The actual value of $\delta_n$ depends on the income of the household concerned, on the partial utilities and prices of all available dwelling types and on the prices of all other consumption goods. This means that we have:

$$\delta_n = \delta_n(y, p, w_i, \ldots, w_N, \pi), \quad n=1,\ldots,N. \quad (18)$$

Since dwelling characteristics are given, as are the prices of other consumption goods, these arguments can be suppressed. Variation in demand among households thus depends in fact only on variations in income and prices. After substitution of (2) we can therefore write $\delta_n - \delta_n(y, \omega)$ instead of the more cumbersome expression in (18), as has already been done.
In the previous section.

In order to derive an expression for the total demand $D_n$ for dwellings of type $n$ we have to know something about the income distribution. We use the following definition:

$$D_n = b_n \int_{-\infty}^{\infty} \delta_n(y, \omega) f(y) \, dy, \quad n=1,...,N. \quad (19)$$

If the set of incomes for which $\sum_n \delta_n$ exceeds 1 is of measure zero, as will usually be the case, we have:

$$\sum_{n=1}^{N} D_n = b_n. \quad (20)$$

On the basis of the assumptions that have been made it is possible to demonstrate existence of an unique equilibrium in the present model if the demand functions are continuous:

**Proposition 1** If assumptions 1...5 are satisfied and the market demand functions are continuous in the allocation variables, there exists an unique equilibrium.

The proof of this proposition has been relegated to the appendix. It should be observed that the proposition does not refer to one specific allocation procedure. In the next four sections we will introduce and discuss four different allocation mechanisms in the model. It is shown in the appendix that the proposition is relevant to all these mechanisms. It is also shown there that, under special circumstances, the condition of continuity can be violated.

Section 5 is devoted to a discussion of the situation in which an equilibrium has to be reached by uniform prices, i.e. prices which are independent of income. In section 6 we introduce the possibility to finance housing costs (partly or completely) by means of a tax. Rent-subsidies are incorporated in section 7. Finally, section 8 is devoted to a discussion of income prices as a means to equilibrate the market. The situation of uniform prices can be identified more or less with allocation by means of...
The other three allocation mechanisms depend on intervention by the government.

5 The validity of the 'iron law'

In the present and next sections we will study the conventional allocation system in which there are uniform prices, i.e. price which are equal for all households, for all available types of dwellings. We start in this section with an examination of the conditions under which the iron law is valid. An assumption that can be interpreted as the exclusion of inferiority of housing turns out to be a sufficient condition for this to be the case. We continue in the next section with a closer examination of the allocation at uniform prices and its normative characteristics.

We start with the simple observation that, under uniform prices, the price of a dwelling type is an increasing function of its partial utility. To see this it suffices to observe that the availability of a dwelling type which has a higher quality and is less expensive than some other dwelling type implies that demand for the latter type will be zero.

It is tempting to go one step further and conjecture that there is also a correlation between the price paid for a dwelling and the income level. This would establish the 'iron law', referred to above, as a theorem. However, it turns out that we need an additional assumption before we will be able to make this second inference. Even in the case where preferences are (weakly) separable in housing characteristics, as was assumed above, the strong relationship between income and housing demand that is implied by the 'iron law' does not necessarily exist. The reason is that an increase in income does not automatically imply a greater willingness to pay for the same increase in dwelling quality.

The situation is illustrated by the three diagrams on the next page. All three figures have the partial utility of housing $w$, and income-net-of-housing-costs on their axes. Three indifference curves are drawn and four combinations $(w, y-p)$ are shown. Two of these refer to a low income level, $y_1$, two to a medium income level, $y_2$, and two to a high income level, $y_3$. There are two dwelling types with partial utilities $w_1$ and $w_2$ and prices $p_1$ and $p_2$ respectively. The price of dwelling 1, which has the higher partial utility, is higher than that of 2, while the household with the medium
Figure 1  Allocation at uniform prices: housing is non-inferior.

Figure 2  Allocation at uniform prices: housing is inferior.

Figure 3  Allocation at uniform prices: the boundary case.
income is indifferent between both. In figure 2 the reverse situation is pictured: the household with the low income prefers the expensive dwelling, the household with the high income the cheap dwelling, while the household with the medium income is again indifferent. Figure 3 shows the boundary case in which all households are indifferent.

The 'iron law' is only valid in the situation pictured in figure 1. The figures suggest that the changes that occur in the curvature of the indifference lines when income rises play a crucial role. In figure 1 the slope of these curves becomes steeper as income rises, in figure 2 the reverse is the case and in figure 3 the slope remains the same. The slope of the indifference curve is given by the following formula:

\[
\frac{\partial w}{\partial (y-p)} = - \frac{\partial v}{\partial (y-p)} \frac{\partial v}{\partial w}. \tag{21}
\]

We may denote the slope of the indifference curve through a point \((w, y-p)\) as a function \(s\), \(s=s(w, y-p)\). It seems to be required, for the iron law to hold, that this slope becomes more negative as \(y\) increases. This conjecture is confirmed by:

**Proposition 2** The 'iron law' is valid if \(\partial s/\partial y > 0\).\(^8\)

**Proof.** Suppose that \(\partial s/\partial y \leq 0\), but that there, nevertheless, is a pair of households \((1, j),\) of which the one with the highest income occupies the dwelling that is placed lowest in the hierarchy. Since prices are not income-dependent this means that the same difference \(w_1-w_2\) in \(w\) can for the lower income be traded off by the income difference \(p_1-p_2\), while this is impossible for the higher income. This gives a contradiction with the condition of the proposition. □

We conclude from proposition 2 that the condition on the slope of the indifference curves implies that households with high incomes will spend more on housing than households with low incomes. It can therefore be interpreted as a statement of the non-inferiority of housing.\(^9\) Since this characteristic is without exception found in empirical research, we will adopt it as an additional assumption:
Assumption 6 Housing is a not inferior, i.e. the utility function is such that:

\[ \frac{\partial s}{\partial y} > 0 \]  

(22)

The 'invisible hand' behind the allocation mechanism studied here is the trade-off between additional disposable income (i.e. income net of housing costs) and a better dwelling (i.e. a dwelling type which is placed higher in the hierarchy). Low income households choose in favour of more disposable income, while high income households are ready to pay the higher price.

The equilibrium under uniform prices can be studied in detail when use is made of the known validity of the 'iron law' which enables us to formulate the following equations, which must be valid in equilibrium:

\[ \int_{y_1}^{y} f(y) \cdot dy = S_1, \quad (23a) \]

\[ \int_{y_n}^{y_{n-1}} f(y) \cdot dy = S_n, \quad (23b) \]

\[ \int_{0}^{y_{N-1}} f(y) \cdot dy = S_N. \quad (23c) \]

In these equations the values \( y_n \) are the lower bounds of the segments of the income distribution that are relevant for the demand for dwellings of type \( n \). These equations therefore allow us to determine these critical income levels recursively.

At a particular income level \( y_n \) we should have:

\[ v(w_n, y_n - p_n) = v(w_{n+1}, y_n - p_{n+1}), \quad (24) \]

\[ n-1, \ldots, N-2. \]

where the superscript 1 refers to the equilibrium under uniform prices. These equations, which can be regarded as equilibrium conditions, allow us
to derive a relationship between the equilibrium prices $p^1_n$ and $p^1_{n+1}$. This relationship is conditional upon the value of $y^*_n$. It will be denoted as follows:

$$p^1_n = g_n(p^1_{n+1}) \quad n=1,\ldots,N-1$$

where $g_n$ is an increasing function.

The significance of the equations (25) is that they enable us to determine the equilibrium prices recursively as functions of $p^*_N$. The total revenues $P$ of the housing stock can now be computed as:

$$P = \sum_{n=1}^{N} g_n(p^1_n) \int_{y^*_n}^{y^*_{n-1}} g_n(y, \varphi) f_n(y) dy, \quad (26)$$

where $y^*_0$ should be identified with $\varphi$ and $y^*_N=0$; $g^n$ denotes $g_n(g_{n+1}(\ldots g_{N-1}(p^1_N)\ldots))$. The total revenues should be equal to the total costs, given in (8). Since the revenues are an increasing function of $p^*_N$, it is easy to find the price $p^*_N$ that at which the budget is balanced. We are therefore able to compute the equilibrium prices in the case of equal sharing. It may be noted that it makes no sense to divide these prices in a part that is identified with rent and another part that is identified with a subsidy or tax. The subsidy or tax can be fixed at an arbitrary value $q$ and the equilibrium rent $r^*_n$ as the difference between the equilibrium price $p^*_n$ and $q$.

If assumption 5 is valid, the only possible way to avoid the 'iron law' is the introduction of prices which are dependent on income. In subsequent sections we will examine several ways to introduce this dependency. For later reference we will denote the equilibrium variables under allocation with uniform prices by means of a superscript 1. We have: $p^1_n = w^1_n = r^1_n$ for all $n$.

6 Financing housing by means of taxation

It has already been mentioned in the first two sections of the paper that many politicians regard the market allocation as unjust. They feel that the allocation of the housing stock, as well as of other commodities, can be
improved if the government takes measures that influence this allocation. For the moment, we will make no attempt to analyze this point of view, but simply note that such measures have been taken in many countries all over the world. We are interested in the effects of such measures on the allocation and will study these them in the present and subsequent sections of the paper. We turn to normative questions in the final section.

Let us, first, adopt the view that housing is a public good, whose costs and benefits have to be distributed over the population in some fair way. It is no longer required that the sum of the rents that have been paid are sufficient to cover the costs associated with housing. Instead, we will now allow for the possibility that a part of the these costs will be financed out of taxes. The amount of tax that is used to finance housing that will be paid by a household with income $y$ will be denoted as $q$, and is dependent on household income $y$ and a parameter $\alpha$:

$$q=q(y, \alpha).$$

The parameter $\alpha$, which may be vector-valued, governs the value of the tax for the various income levels. It will be assumed that the tax is always nonnegative, that it is a non-decreasing function of income, and that the marginal rate of taxation is less than 1.

The value of the parameter $\alpha$ determines the total revenues $Q$ from tax-paying that will be used to finance housing costs. It may happen that different configurations of $\alpha$ give the same total revenues. In that case the value of $\alpha$ that is considered to be the most fair one may be chosen.

The feasibility of the procedure outlined above is investigated in the appendix. It is shown there that for any given value of $\alpha$ there exist equilibrium rents, which will be referred to as $r^2_n$, $n=1...N$, at which the government budget is exactly balanced. These rents are equal to the allocation variables $\omega^2_n$, but are different from the variables $p^2_n$. We will discuss this equilibrium in the remainder of the present section.

The main similarity between the allocation at uniform prices and the one considered here is that the distribution of the households over the housing stock is the same. The allocation system considered in the present section can be considered as a two-stage procedure. In the first stage all households have to pay a tax, which is dependent on their income, while in
the second stage the housing stock is allocated by means of uniform prices. The effect of the tax is that the income distribution has changed and that the amount of rent revenues needed to balance the government budget is now equal to \( C - Q \) instead of \( C \). The allocation of the housing stock itself remains unchanged. A household that occupied a dwelling of type \( n \) under an allocation system with uniform prices will still occupy such a dwelling after the introduction of tax-financing. The 'iron law' will therefore still be valid.

It follows immediately that equations (25) are still valid in the present context. However, instead of equation (26) we will now have the following equations:

\[
\forall (w_n, y_n, q(y_n, \alpha)) = \forall (w_{n+1}, y_n, q(y_n, \alpha)), \quad (28)
\]

which can be used to determine recursive equations for the equilibrium rents \( \tau_n^* \) for any given way of tax levying. The equations are of course analogous (but not identical) to (24).

Let us now examine the question whether the prices \( p_n \) in the present situation can be the same as under uniform pricing. The answer is that this will be impossible as soon as the tax \( q \) shows some variation with income, i.e. if \( q(y_n, \alpha) > q(y_{n+1}, \alpha) \) for some \( n=1,...,N-2 \), where \( y_n \) is the critical income level for dwellings of type \( n \) introduced in the previous section. As soon as this variation exists the equations (30) can no longer be all identical to (26). On the other hand, it is obvious that the same equilibrium as under uniform prices can be reached when the tax \( q \) is the same at all income levels. It can therefore be concluded that the equilibrium prices \( p^1_n \) will be different from \( p^2_n \) for at least some \( n \in \{1,...,N\} \) if and only if the tax \( q \) does vary with income.

If prices are not the same in both equilibria, some must be lower and others higher, since the net amount of money raised by the tax and the rents must in both situations be equal. It would probably be judged to be desirable if for the lower incomes the price of housing has decreased. We will show that this will indeed be the case.

Assume that in the equilibrium with a housing tax \( p^2_N(y_{N-1}) > p^1_N(y_{N-1}) \), i.e. at the critical income level at which the switch from dwellings of type \( N \) to dwellings of type \( N-1 \) occurs, the dwellings of type \( N \) have become more expensive. Since the partial utilities \( v \) are increasing in income-net-of-
housing-costs, it must be concluded that, because of the equality sign in (29), $p^2_{N-1}(y_{N-1})$ also exceeds $p^1_{N-1}(y_{N-1})$. Since $q$ is non-decreasing in income, we must conclude that $p^2_{N-1}(y_{N-2})$ will also be greater than $p^1_{N-1}(y_{N-2})$. Now we can repeat the same reasoning until we find that $p^1_1(y_1) > p^1_1(y_1)$. This gives a contradiction with the condition that the budget of the government will be balanced in both equilibria. We must therefore conclude that a housing tax that is non-decreasing in income leads to a lower price for the worst type of dwelling. Analogously it follows that the price for the best types of dwelling increases. (It should be observed that the discussion above amounts to the 'full' price, viz. the sum of the rent and the housing tax. We cannot exclude the situation in which the rents, even for the highest preferred types of dwellings will be lower after the introduction of the housing tax.) This can be summarized as follows:

$$p^2_1 > p^1_1 \quad (29a)$$

$$p^2_N > p^1_N \quad (29b)$$

With respect to the differences in housing prices it can be observed from assumption 5 that a lower income gives rise to a smaller willingness to pay for a given improvement in dwelling quality. The introduction of the tax implies that the incomes which are relevant for the determination of the rents, viz. the incomes-net-of-taxes have all decreased. This implies that the differences between the rents of all types of dwellings will decrease, i.e.:

$$p^1_n - p^1_{n'} > p^2_n - p^2_{n'} \quad (30)$$

$$n, n'=1...N, n'>n.$$  

This implies a tendency towards greater equality in rents, as a consequence of the introduction of the housing tax.

7 Tax-Financing and Rent Subsidies

In the present section we introduce a rent subsidy which can be regarded as a simplified version of the actual system of rent subsidies used in the Netherlands. The government subsidizes rent expenditures of households when they exceed a predetermined share of their budget. This implies that the rents $r_n$ now also become income-dependent. In principle there is a uniform
rent \( \omega_n \) for each type of dwelling, but the actual rent may differ:

\[
\begin{align*}
\eta_n(y) &= \omega_n - \max \left\{ 0, \sigma_1 \cdot (\omega_n - \sigma_2 \cdot y) \right\} \\
&\quad \text{for } n = 1, \ldots, N.
\end{align*}
\]

It will be assumed throughout that \( 0 < \sigma_1, \sigma_2 < 1 \).

The revenues from rent will in this case be equal to:

\[
R = \sum_{n=1}^{N} \int \left[ \rho_n - \max\left\{ 0, \sigma_1 \cdot (\rho_n - \sigma_2 \cdot y) \right\} \right] f_n(y) \, dy,
\]

which will, of course, never exceed the revenues associated with the same rents without subsidies. This implies that the introduction of a rent subsidy forces the government to increase the rents or the taxes (or both). It will be assumed throughout that the subsidy is financed purely from higher taxes.

The rent subsidy is intended to help the poor. It will only be effective in this respect, however, if households with a low income spend a larger share of their budget on housing than households with a high income, i.e. when housing is a normal good and not a luxury good. This will be assumed to be the case throughout this section.

The effects of the introduction of the rent subsidy are complex. The first consequence is, of course, that households who spend more than the predetermined share of income on housing receive a subsidy. One may wonder whether these households are always the poorest ones. In order to get a first impression of the answer, let us return to the situation analyzed in
Figure 5 Effects of a Rent Subsidy.

the previous section. In the equilibrium situation the iron law is valid and the households with the lowest incomes live in the cheapest dwellings. The expenditures on rent will increase with income. However, the share of the rent in total income does not necessarily increase. This share will first decrease, until a switch is made to a better, and therefore more expensive, type of dwelling and will then again decrease. The picture is illustrated in figure 4. It is shown there that it may happen that also relatively rich households will be subsidized, even though the share of the rent in total income is in general declining, while there may also be some relatively poor households who will not receive a rent subsidy.

It should be noticed, however, that the introduction of a rent subsidy will change the choice behaviour of the households. They become inclined to switch to a dwelling of a higher quality at a lower income level, because the rent subsidy enables them to bear the burden of the higher costs earlier. This implies that the critical income levels will change. When the subsidies have a marginal effect only, i.e. when switches in the demand of households from type n+1 to type n only, households with an income just below a critical income level \( y_n \) will now also demand a dwelling of type n. It must therefore be expected that - at the initial equilibrium prices - some or all critical income levels \( y_n, n=1...N-1 \) decrease. Since this is clearly incompatible with equilibrium, there must be a counteracting relative increase in the prices of the dwelling types for which demand shows a net increase and a relative decrease in the prices of the dwelling types for which demand shows a net decrease. Although the effects cannot be determined in precise terms, it may be said that there will be a decrease
in the price of at least one dwelling type \( n \) and and increase in the price of at least one dwelling type \( n' \), \( n' < n \), while the prices of the dwelling types \( n'' \) inbetween these two may either increase or decrease.

However, we cannot be sure that the effects of the introduction of a rent subsidy are marginal only. What can be said is that the ultimate effect of the subsidy on the allocation of the households over the housing stock will be nil, unless the ‘iron law’ is violated. Under the present allocation mechanism this is possible. The reason is that the subsidy introduces differences in the prices that various households have to pay. The introduction of the rent subsidy changes the additional amount of money that has to be paid for a given increase in quality. It may therefore happen that at income level \( y' \), where no housing subsidy will be received, a household prefers the cheaper dwelling of type \( n \) to the more expensive one of type \( n' \), while at the lower income \( y'' \) the reverse is the case. The situation is illustrated in figure 5. The indifference curves in this figure are the same as those drawn in figure 1. However, in figure 5 dwellings of type 2 are subsidized to such an extent that the household with the low income prefers the high quality, even though the household with the medium income chooses in favour of the lower rent. As a consequence of the change in demand the price of the high-quality dwelling will increase and that of the low-quality dwelling decrease and one can imagine that in the new equilibrium the household with the medium income will now choose the low-quality dwelling, while both the low and the high-income households prefer the high-quality dwelling. Whether or not the ‘iron law’ will become violated as a consequence of the introduction of the rent subsidy depends on the preferences of the households and on the parameters \( \sigma_1 \) and \( \sigma_2 \).

In order to analyze the possible violation of the iron law, observe that the rent subsidy reduces the rent difference between two types of dwellings \( n \) and \( n' \) for which subsidy can be obtained from \( \omega_n - \omega_{n'} \) to \( \sigma_1 (\omega_n - \omega_{n'}) \). As long as income does not change too much, and for both dwelling types a subsidy can be obtained, the rent difference remains the same. Rent differences between dwelling types that are not subsidized will of course also remain the same. What changes, however, is the difference between rents of dwelling types which are subsidized and dwelling types which are not subsidized. The subsidized dwelling types all become more expensive, while the others keep the same rent. When large changes in income occur,
some dwelling types for which originally a subsidy could be obtained will now take a share of income which is less than \( \sigma_2 \) and will no longer be subsidized.

In the equilibrium under the present allocation mechanism it is always possible to distinguish a finite number (possibly zero, but this is an uninteresting case) of income intervals for which a rent subsidy is received. Since the rent differences between the subsidized dwellings on each interval are the same, it may be concluded that on each such interval the 'iron law' will be valid. That is, if there occurs a shift from one subsidized dwelling type to another the type that is chosen at the highest income levels is the most expensive and has the highest quality. The same is of course true for the intervals of incomes at which no subsidy is obtained.

Violation of the 'iron law' is therefore only possible at incomes where subsidized and non-subsidized intervals touch each other. At such points demand may shift from a dwelling of type \( n \) to one of type \( n' \), with \( n' > n \). As a typical pattern one would of course expect that the poor will all receive an income subsidy, while the rich will not. It should be noted, however, that this pattern is not automatically implied by our model.

Let us first analyze what is necessary for a backslide to occur. The increase in the price difference between a subsidized and a non-subsidized dwelling equals \( \sigma_1 \cdot \sigma_2 \cdot (\Delta y) \). The indifference curves become less steep when income rises, which implies that the household becomes willing to pay more for the same increase in quality. For a backslide to occur, it is therefore necessary that the flattening of the indifference curves is insufficient to compensate for the increased price difference. A backslide is therefore only possible when \( \partial s / \partial y \) is smaller than \( \sigma_1 \cdot \sigma_2 \). Note that this analysis implies that a backslide will never occur when \( \partial s / \partial y > 1 \), i.e. when housing is a luxury good.

Let us now see what is necessary for a reswitching from no subsidy to subsidy to occur. The increase in the price difference, again, equals \( \sigma_1 \cdot \sigma_2 \cdot \Delta y \). The flattening of the indifference curves should now be sufficient to compensate for the increased price difference. In order to exclude reswitching we should therefore assume \( \partial s / \partial y \) to be larger than \( \sigma_1 \cdot \sigma_2 \). It follows also that no reswitching occurs whenever \( \partial s / \partial y > 1 \), which implies that housing is a luxury commodity. We must therefore conclude that the introduction of a rent subsidy will always favour the rich households.
when housing is a luxury good.

The typical pattern one expects is, of course, that that in which only poor households are subsidized. In order to study a special case in which such a pattern can occur, we make the following assumption:

\[ \frac{\partial s}{\partial y} = c \]  

(33)

where \( c \) is a positive constant.

When \( c \) is smaller than 1, housing is a normal good. When \( c > \sigma_1 \sigma_2 \) there can be no backslide, only reswitching. When \( c = \sigma_1 \sigma_2 \) the demand functions are not continuous, and an equilibrium may not exist. The most relevant case occurs when \( c < \sigma_1 \sigma_2 \). In that case there may be a backslide, although this is not necessarily the case. The quality of the dwelling occupied by households still shows a tendency to rise with the income level, but there is one possible discontinuity in this relationship.

A final effect of the introduction of the housing subsidy that has to be noticed is that the housing tax must be increased in order to finance the subsidy. This will probably done in such a way that the rich households experience a heavier tax increase than the poor ones. The result will be a mitigation of the differences between the rents of the dwelling types and this counteracts the price increase for high quality dwellings which should be expected as a consequence of the introduction of rent subsidy. It will therefore compensate these households to some extent for this price increase.

8 Income Prices

The final allocation mechanism we want to consider makes use of income prices. In this situation the prices \( p_n \) are assumed to be set as a fraction of household income:

\[ p_n(y) = \omega_n y \]  

(34)

\[ n=1,\ldots,N. \]

For the iron law to be valid under this allocation mechanism, it is
necessary that the trade-off between a given increase in dwelling quality and a given additional share of income to be spent on housing should change in favour of the former when income rises. It will be clear that this requires a stronger condition on the slope of the indifference curves than that embodied in assumption 5.

Let us consider the situation in which a household can choose between a dwelling of type \( n \) and one of type \( n' \). The associated change in dwelling quality is \( w_n - w_{n'} \), \( w_n > w_{n'} \), the associated change in income \( (w_n - w_{n'})y \). If the more expensive dwelling will be chosen at the higher income and the cheaper one at the lower income the slope of the indifference curves should flatten at a rate \( ds/dy > 1 \). This implies that housing should be a luxury good, in order to ensure that the iron law will still be valid.

It will be convenient to rely again on assumption 7. If \( c < 1 \) the iron law will be exactly reversed. The willingness to pay for a given increase in dwelling quality rises, but this rise is not sufficient to compensate for the associated rise in the price. The implication is that there will, again, be a perfect correlation between income and housing quality, but now in such a way that a higher income is associated with a lower quality of the dwelling that will be occupied.

If \( ds/dy \) is between zero and one, but not constant, the iron law may still be valid in special cases. But very different situations, in which it is hard to discover any relation between income and housing, may also occur. For instance it may happen that the iron law is reversed initially, but is valid for higher incomes, while ultimately it will be reversed again.

9 Which distribution is equitable?

In the present section we will try to pursue the analysis of the preceding ones a little bit further by posing the question why governments undertake various measures in order to influence the allocation on the housing market. It has already been mentioned in the introduction that the Pareto-criterion will be of little help in this respect.

A possible alternative is the utilitarian solution in which the sum of the individual utilities is maximized. This approach has been used in the theory of optimal taxation (see Mirrlees [1971] and Atkinson and Stiglitz [1980, ch. 13, 14] for an overview). In the present partial model we do not
consider the relation between income generation and taxation, and this is
doubtless one of the reasons of the unrealistic results. When the utility
function is concave, which seems to be a normal requirement (especially
when one adopts a cardinal concept of utility), redistribution of utility
from the wealthy to the less wealthy will always be beneficial. The
implication is that prices are such that in the optimal situation the
remaining differences in income are just sufficient to compensate for the
unavoidable differences in housing quality. This implication does not seem
to be realistic.

We will adopt the theory of fairness as an alternative. This theory
originates from Foley [1967] and has been developed by Kolm [1972] and
Varian [1974]. For reviews we refer to Thomson and Varian [1985] and Baumol
[1986].

The central concept in this theory of distributional justice is envy. A
particular distribution of a given amount of commodities gives rise to envy
when some of the people engaged in the distribution would prefer to have
received the amount that others get to their own share. When the
distribution concerns a homogeneous commodity, equal division is the only
fair solution. When the commodity to be distributed is heterogeneous, other
solutions than equal division may exist and be preferred to that one. There
may be a best solution among a number of possible fair solutions.

In the present situation we will regard the housing stock as the
heterogeneous commodity that has to be distributed. Since housing is not
only heterogeneous, but also indivisible, the problem cannot be solved by
equal division. It is, in fact, easy to see that there will be no solution
to our problem if we consider only the distribution of the housing stock.
Since all consumers have the same tastes, those allocated to a dwelling of
type 1 will be envied by all others. However, we have to be aware of the
fact that not only housing itself, but also the costs associated with
housing have to be distributed. If the households occupying a dwelling of a
higher quality would have to pay a much larger share of these costs, they
will probably no longer be envied. \(^\text{11}\)

Let us now reconsider the allocation at uniform prices. In equilibrium
all households can make their utility maximizing choice out of the \(N\)
combinations \((w_n, p_n^1)\). The price should now be viewed as the share the
households pays in the total costs associated with housing. This
interpretation is facilitated by the fact that in equilibrium the budget condition of the government is exactly satisfied. We have to conclude therefore that the distribution of housing stock and housing costs that corresponds with uniform prices is fair, in the sense given to this term by Foley c.s.

It should, of course, be realized that this fairness has a limited meaning. It is conditional upon the distribution of income, which is taken to be exogenously given. In the literature the term 'incremental fairness' is sometimes used to indicate the limited sense of the term in contexts such as the present one.\textsuperscript{12} We have reached the following conclusion:

\textbf{Proposition 3} The allocation at uniform prices is incrementally fair.

The allocation at uniform prices turns out to be the only one that is incrementally fair. As soon as a tax is introduced which is income-dependent, rich households will envy the lower total price of housing that poor households pay, although they do not envy the housing of the poor as such. When also a rent subsidy is introduced and the 'iron law' is violated, some households who are not subsidized will envy the better housing situation of some poorer households as well. Allocation by means of income prices will also always cause envy because of the lower housing costs of the poor and envy because of the better housing situation as soon as the iron law is violated. It must be concluded therefore that the theory of (incremental) fairness is not of much help for the determination of an equitable allocation mechanism for the housing market.

The foregoing suggests that the actual judgements about the desirability of policy measures that influence the housing market do not accept the income distribution as given. More useful criteria for judging housing market policy will probably have redistribution of income as an explicit goal. However, the incorporation of such a target introduces new problems, since income is redistributed by other government measures as well, and one would probably like to coordinate the various means of redistribution. The possible negative effects of such a redistribution should then also be taken into account. Partial models, such as the one developed in this paper, would then be inadequate.

In actual policy-making, however, the housing market policy is usually
considered as a special area, which has no direct links with the department of social affairs. The suggestion that this area should only be viewed as a specialized subfield of the broader area concerned will probably not be welcomed by politicians. One must therefore conclude that a welfare-economic analysis of housing market policy measures is fraught with paradoxes, if not contradictions. The normative criteria that we possess do not support measures that introduce income-dependent prices as long as the income distribution is taken as given (incremental fairness), or support an extensive redistribution (utilitarianism). But if income redistribution is regarded as a major goal of housing market policy one must pose the question whether how such a policy should be coordinated with other measures intended to reach this goal and whether such a policy would provide a good means to pursue it.

There may be another approach to analyze housing policy. Instead of a normative approach, one may adopt a positive one and ask whether the introduction of policy measures that influence the allocation on the housing market are attractive to introduce for politicians. A sceptical observer may remark that in political matters votes are the only thing that counts. Measures that redistribute income by taking from a relatively small number of rich and giving to a relatively large number of poor will therefore always be attractive, especially for politicians who recruit their voters from the relatively poor (see Aumann and Kurz [1977]). This vision has some attractive characteristics. It was no accident that the rent subsidy was introduced by the most left-wing government the Netherlands has ever experienced. However, it also has its clear limitations. For instance, it leads one to expect that, assuming democracy, a relatively unequal distribution of income would tend to be correlated with more measures intended to redistribute income than a relatively equal one, while reality suggests the exact opposite. Moreover, the opinion that the income distribution resulting from a free functioning of the markets forces needs correction is widespread in many countries and not confined to opportunistic politicians and poor households that envy rich ones. But this takes us back, once again, to the normative approach.

11 Conclusion

In this paper we have analysed the effects of a number of policy measures, intended to improve the allocation on the housing market, in a
simple model. This model can be viewed as a pure situation in which the relation between income distribution and allocation on the housing market can be studied.

It has been shown that the 'iron law' is valid under uniform pricing when housing is a non-inferior commodity. When the costs of housing are financed partly by means of tax payments the allocation remains the same, although the distribution of the costs of housing changes somewhat. Introduction of rent-subsidy or income prices may cause violations of the iron law. However, it turns out to be hard to predict the exact pattern of the allocation implied by these alternative allocation mechanisms, except when housing is a luxury good. In that case the iron law will always be valid.

It is hard to judge whether or not the introduction of allocation systems different from uniform prices results in an improvement of welfare. The conventional toolbox of welfare economics, including the theory of fairness which has been developed recently, does not contain instruments which are helpful in this respect. The vague notion that the income distribution is not what it should be and that therefore the same will be true for the distribution of the housing stock seems to be a weak basis for policy measures. It must be concluded therefore that the normative aspects, which are of utmost importance as a justification for these measures turn out to be hard to analyze. The positive analysis carried out in the preceding sections may nevertheless be useful in that it gives one the possibility to ask politicians whether this is what they had in mind.
Notes

1 The concept of housing needs can perhaps be considered as analogous to that of poverty (see e.g. Hagenaars [1986]).
2 The paradoxical aspects of this widely, if not universally, held opinion may be noticed: one can hardly imagine a society in which all men are equal and one can hardly give content to the concept of equality without reference to a society in which it should be used.
3 See e.g. Sen [1987, pp.35-38] where the changes in the initial position that may be necessary to arrive at an optimal situation are compared with those realized after a revolution.
4 I.e. preferences must be such that the occupation of two dwellings will always result in a lower utility than occupation of one dwelling.
5 This assumption is equivalent with the requirement that \( u' \) should be increasing in \( w \) and \( x \). It is simply a matter of convenience that it is introduced here in this alternative form.
6 It should be noted that the possibility that this set is of positive measure situation cannot be excluded by simply assuming (as we have done) that the income distribution is such that the set of households earning an arbitrary income \( y \) is of measure zero. Tastes and the way prices depend on income can also be the cause of a situation in which the set of households being indifferent between two or more dwelling types has a positive measure. Such situations correspond with a violation of the continuity condition of of the demand functions, required in proposition 1.
7 Assumption 6 can be rewritten (making use of eq. 4) as:

\[
\frac{\partial v}{\partial (y-p)} > \frac{\partial^2 v}{\partial^2 w (y-p)}.
\]

It is common to assume (as has been done in assumption 3) that both \( \frac{\partial v}{\partial w} \) and \( \frac{\partial v}{\partial (y-p)} \) are positive and that \( \frac{\partial^2 v}{\partial (y-p)^2} \) is negative. We can then be sure that assumption 2 is satisfied when \( \frac{\partial^2 v}{\partial w \partial (y-p)} \leq 0 \). It should be noted, however, that this is not a necessary, but only a
sufficient condition. (It may be interpreted as saying that housing and income to be spent on other consumption goods are substitutes.)

8 One may wonder whether this condition is also necessary. The answer is that in actual situations, with a limited number of existing dwelling types, it is not. It may then be the case that the condition is violated for many combinations of \( w \) and \( y-p \) and that the iron law, nevertheless, holds. However, if we wish to exclude the possibility that the iron law will be violated for any possible pair of dwellings, then the condition is also close to necessary. This can be seen by considering the situation in which the inequality in the condition is reversed for a measurable set of combinations of partial dwelling utilities and incomes-net-of prices. It is then possible to find a combination \((w, p)\) for which the iron law does not hold.

9 One may wonder whether this definition corresponds with the usual notion of non-inferiority, viz. the requirement that the demand function is non-decreasing in income. It can be shown easily that in the case of two commodities the requirement that \( \frac{\partial s}{\partial x_1} > 0 \), with \( s \) the slope of the indifference curve and \( x_1 \) one of the two commodities, rules out the possibility that the other is inferior. In the general, \( n \) commodity, case, one may study the demand for good \( n \) by considering all other commodities as a composite good and it can be verified that the requirement \( \frac{\partial s}{\partial c} > 0 \), where \( c \) is the composite good, rules out the possibility that \( x_n \) is inferior. The conjectured correspondence therefore exists.

10 The actual subsidy that is used in the Netherlands is more complicated than the one presented here. There is a maximum amount of rent that can be subsidized and the fraction \( \sigma_2 \) rises with income. This makes it impossible that the subsidy is only beneficial to the rich.

11 This procedure is analogous to the one proposed by Kolm [1972, pp. 67-71] for fair division in the presence of indivisibilities in that the distribution of an indivisible good is made easier by considering it simultaneously with something perfectly divisible. In Kolm's case a positive amount of money, in our case the costs of housing. See also Crawford and Heller [1979], especially sections 1 and 2.

12 We ignore here the problems that can be associated with the concept of incremental fairness, see Feldman and Kirman [1974].
13 It should, however, be noted that an alternative function of this subsidy is to help households who are unable to find a dwelling with a reasonable rent on the heavily disequilibrated Dutch housing market. The model used here is of course unable to analyze the effectiveness of the measure in this respect.

14 It may, however, be hard to determine which of the two related phenomena is the cause and which the effect.

15 The foregoing suggests that opportunistic politicians support measures that redistribute incomes from rich to poor households and use the widespread value judgement concerning the equitable distribution of the housing market in order to increase the support for such measures.
References

Atkinson, A.E., and J. Stiglitz [1980] Lectures in Public Economics,
Appendix  Existence of equilibrium

All Two Lemma's

We will first prove two characteristics of the market demand functions which will be of use for the existence proof that will be given later on:

Lemma 1 The N dwelling types are weak gross substitutes when assumption 3 is satisfied.

Weak gross substitutability means that \( D_n \) will be a non-increasing function of its own price \( p_n \) and a non-decreasing function of all other prices \( p_{n'} \), \( n' \neq n \). In the present context it can be interpreted as saying that demands \( D_n \) are non-increasing in \( \omega_n \) and non-decreasing in \( \omega_{n'} \), \( n' \neq n \).

Proof. Assume that \( \omega_n \) increases. Then the price \( p_n(y) \) increases and the utility \( v_n \) decreases. As a consequence, utility maximizing households who demanded a dwelling of type \( n \) before the price increase may decide to switch their demand to another type of dwelling. Households who originally demanded another type of dwelling will never decide to switch their demand to a dwelling of type \( n \).

The consequence of a decrease in \( \omega_n \) are opposite. □

Two remarks are in order:

1. Gross substitutability of the demand function is, in the present context, an almost immediate consequence of utility maximization. It has been derived for a similar model by Sweeney [1974].

2. Although it has been assumed that the indirect utilities \( v_n \) are (strictly) increasing in income-net-of-housing-cost \( y-p_n \), market demand cannot be shown to satisfy strong gross substitutability. The reason is that the effects of a change in a price \( p_n \) may be zero (when \( D_n \) equals either 0 or \( b \)) or may be limited to a small number of other demands only.

A second property of aggregate demand that is very helpful for the existence proof is that \( D_n \) becomes close to (or equal to) zero for high values of \( \omega_n \) and close to (or equal to) \( b \) for low (i.e., negative) values of this variable. This property is ensured by assumption 4:
Lemma 2 When assumption 3 is satisfied and $\omega_n$ is fixed for all $n' \neq n$:

\[
\lim_{n \to \infty} D_n = 0, \quad (A1a)
\]

\[
\lim_{n \to \infty} D_n = b, \quad (A1b)
\]

The proof is obvious (use assumption 4).

A third property of the demand functions that is useful for existence proofs is continuity in the allocation variables. In order to see whether this property can be guaranteed, we repeat the definition of market demand:

\[
D_n = b - \sum_{n=1}^{N} \delta_n(\omega, y)f(y)dy, \quad (A2)
\]

We know by assumption 4 that $D_n$ will ultimately become equal to zero when $\omega_n$ becomes infinitely large and will ultimately become equal to $b$ when $\omega_n$ becomes infinitely small. A necessary condition for continuity is that $D_n$ should take on any value between these extremes for finite values of $\omega_n$. Assume that $0 < D_n < b$ at some values of the allocation variables. It follows that there should be some $n'$, $n' \neq n$, for which the expression $v_n - v_{n'}$ changes sign when income varies between $y_{min}$ and $y_{max}$. There exist specifications of $v$ which do not satisfy this requirement under some allocation mechanisms. Continuity of the demand functions can therefore not be ensured in general. Since conditions that guarantee continuity vary with the prevailing allocation mechanism it is not useful to state them at the present level of generality. We will therefore first prove our general existence result, conditional upon continuity, and return to the question whether this property can be guaranteed under the various allocation mechanisms in the final section of this appendix.

A2 Proof of proposition 1

Before proving the proposition itself, we will first prove two additional lemma's:
Lemma 3 If the demand functions are continuous, there exists a vector of allocation variables for which demand equals supply for all dwelling types. This vector is unique up to the choice of a value of \( w_N \) for which
\[
p(\omega_N', y_{\min}) < y_{\min}.
\]

Proof. We start with fixing all allocation variables at an arbitrary value \( \omega_N' \). Because of the existing hierarchy of dwelling types, all households will demand a dwelling of type 1. Now increase \( \omega_1 \), and therefore all prices \( p_i(\omega_1, y) \), until the demand \( D_1 \) becomes equal to supply. Assumption 3 and the continuity of the demand functions ensure that this is possible.

In the situation that results all households that do not demand a dwelling of type 1 demand one of type 2. Therefore the price \( p_2 \) should be increased until this excess-demand is also removed. During this price increase care has to be taken that demand for dwellings of type 1 remains equal to supply; this can be done by further increases in the price \( p_1 \).

This procedure can be continued until all excess demand is removed. The final situation is clearly a price equilibrium and has the price \( p_N \) equal to the value that has been chosen before the start of the procedure. (A more formal description of such a price adjustment process in a general equilibrium context can be found in Van der Laan and Talman [1987].)

To show the uniqueness, assume that there are multiple equilibria. Let \( \omega(1) \) and \( \omega(2) \) be two such equilibria, with \( \omega_N(1) = \omega_N(2) \). There should be at least one \( n \) for which \( \omega_n(1) > \omega_n(2) \). Assume, without loss of generality, that \( \omega_n(1) < \omega_n(2) \). Let \( A \) be the set of dwelling types \( n' \) with \( \omega_n(1) < \omega_n(2) \) and let \( D = \sum_{n' \in A} D_n \), the total demand for dwelling types whose prices have increased. In both equilibria \( D \) should take on the same value.

Now consider an individual who chooses a dwelling of type \( n'' \in A \) in equilibrium 1. Such a household will certainly not choose a dwelling of a type \( n'' \in A \) in equilibrium 2. It follows that all households choosing of a dwelling type \( n'' \in A \) in equilibrium 1 should continue to choose for such a dwelling type in equilibrium 2. However, there must be at least one income level \( y_\star \) at which \( v_n(1) = v_n(2) \) for some \( n'' \in A \) and \( n'' \in A \). If the prices \( p_{n''}, n'' \in A \), increase and all other prices \( p_{n''}, n'' \in A \), decrease or remain the same, \( v_n(2) < v_n(1) \). This implies that there must be some individuals who switch
from a dwelling of type $n' \in A$ to one of type $n \notin A$. But this contradicts our result that $D_A$ should be identical in both situations. □

Because of the uniqueness, we can write the equilibrium values of the allocation variables $\omega_n$, $n=1,\ldots,N-1$, as functions of $\omega_N$ and have:

Lemma 4:

$$\lim_{\omega_N \to \omega^*} \omega_n(\omega_N) = -\infty,$$

$$n=1\ldots N-1,$$

Proof. Assume that the equilibrium value of $\omega_n$ may remain greater than some finite limit value, say $\omega^*_n$, if $\omega_N \to \infty$. By assumption 4, $v(\omega_n', y-p(\omega_n, y)) > \lim_{\omega_N \to \omega^*} \omega_n(\omega_N, y-p(\omega_N, y))$, and this implies that dwellings of type $n$ will not be demanded in equilibria with a sufficiently small (negative) value of $\omega_N$. Hence, we find a contradiction.

The opposite assumption can be shown to be contradictory in a completely analogous way. □

We are now in a position to prove proposition 1:

Proposition 1 If assumptions 1-4 are fulfilled and the demand functions are continuous in the allocation variables, there exists an unique equilibrium.

Proof It should be observed that the values of $P$ that correspond with price equilibria can be viewed as a correspondence of $\omega_N$. It follows from the lemma that these revenues become larger if $\omega_N$ increases and will, by assumption 3, reach or exceed the value $C$ for some finite value. Furthermore, it follows from lemma 4 that the revenues become infinitely small (i.e. negative) if $\omega_N$ decreases without a lower bound. Inbetween there should be at least one value of $\omega_N$ at which the budget can be balanced.

Uniqueness follows from the fact that there is an unique price equilibrium for every value of $\omega_N$. □
A3 Continuity of the demand functions

We now return to the question of continuity. The easiest case occurs when prices are uniform. For this case we have:

Lemma 5 If assumption 6 is fulfilled,

a) the demand functions are continuous when prices are uniform and also when rents are uniform and housing costs are partly financed by taxes,

b) the demand functions are continuous when there is rent subsidy and \( \frac{\partial s}{\partial y} = \sigma_1 \sigma_2 \), possibly except for a set of measure zero,

c) the demand functions are continuous when there are income prices and \( \frac{\partial s}{\partial y} = 1 \), possibly except for a set of measure zero.

Proof. A discontinuity in the total demand \( D_n \) implies that a small increase in the price \( p_n \) can give rise to a 'jump' in this demand. This can only occur if demand switches from \( n \) to (or from) some \( n' \), on at least one interval \([y',y'']\), \( y'' > y' \). This implies that for incomes in this interval, the trade-off between dwelling utilities \( w_n \) and \( w_{n'} \) and incomes-net-of-housing-costs \( y-p_n \) and \( y-p_{n'} \), is exactly the same. But under uniform prices or uniform rents this can never occur when \( \frac{\partial s}{\partial y} > 0 \) everywhere, as stated in assumption 6. We have therefore found a contradiction.

When there is a rent subsidy, the trade-off between quality and price of two dwelling types can be exactly the same for all incomes on a non-trivial interval if one dwelling type is subsidized and the other is not and \( \frac{\partial s}{\partial y} = \sigma_1 \sigma_2 \) on that interval. It follows from assumption 6 that there cannot be a constant trade-off for two dwelling types which are both subsidized or none of which is subsidized.

When there are income prices, the trade-off between price and quality of two dwelling types can only be constant if the household is willing to offer a given fraction of its income for a given increase in quality, i.e. if \( \frac{\partial s}{\partial y} = 1 \) on a non-trivial interval.
<table>
<thead>
<tr>
<th>Year</th>
<th>Authors</th>
<th>Title</th>
</tr>
</thead>
<tbody>
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<td>1988-1</td>
<td>H. Visser</td>
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</tr>
<tr>
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</tr>
<tr>
<td>1988-31</td>
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</tr>
<tr>
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<td>J. Rouwendal, F. Nijkamp</td>
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</tr>
<tr>
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<tr>
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<tr>
<td>1988-36</td>
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</tr>
<tr>
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<td>1988-38</td>
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</tr>
</tbody>
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