RECURSIVE APPROXIMATE MAXIMUM LIKELIHOOD ESTIMATION FOR A CLASS OF COUNTING PROCESS MODELS

Peter Spreij

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by
Peter Spreij
Free University, Amsterdam

ABSTRACT

In this paper we present a recursive algorithm that produces estimators of an unknown parameter that occurs in the intensity of a counting process. The estimators can be considered as approximations of the maximum likelihood estimator. We prove consistency of the estimators and derive their asymptotic distribution by using Lyapunov functions and weak convergence for martingales. The conditions that we impose in order to prove our results are similar to those in papers on (quasi) least squares estimation.
INTRODUCTION

We assume that we are given a complete probability space \((\Omega, F, P)\) together with a filtration \(\{F_t\}_{t \geq 0}\), satisfying the usual conditions in the sense of [2]. All stochastic processes to be encountered below are assumed to be adapted with respect to the given filtration, unless stated otherwise. Similarly the martingale property is also to be understood with respect to this filtration. Let \(N: \Omega \times [0, \infty) \to \mathbb{N}\) be a counting process, such that its Doob-Meyer decomposition takes the following form (in differential notation)

\[ dN_t = \varphi_t^\top \theta dt + dm_t \]  

Here \(\varphi: \Omega \times [0, \infty) \to \mathbb{R}^{d}\) is a predictable process, \(\theta \in (0, \infty)^d\) an unknown parameter and \(m: \Omega \times [0, \infty) \to \mathbb{R}\) a local martingale. Superscript \(T\) usually denotes transposition.

The purpose of this paper is to give a recursive scheme that generates estimators \(\hat{\theta}\) of the unknown \(\theta\). This scheme is given below as the set of equations (2a-2e). In an earlier paper [6] we have presented a similar but slightly different set of equations. For a heuristic derivation of these equations we refer to [6], where also an account for the terminology approximate maximum likelihood (AML) estimation can be found.

The conditions that we impose in theorem 3 in order to prove a.s. convergence of the estimators are of the same form as those in e.g. [1] and [5], where (quasi) least squares estimation has been studied and considerably weaker than those in [6]. However, we do not need all the conditions of [1]. In the sequel \(\theta_0\) will denote the "true" parameter value, \(\mathbf{1}\) is the vector in \(\mathbb{R}^d\) whose entries are all equal to 1. After giving an assumption on the parameter space, we present our estimation algorithm and an analysis of its asymptotic properties.
THE RESULTS

ASSUMPTION 1: \( \theta_0 \) lies in a compact subset of \( \mathbb{R}^d \). Hence there exists \( \epsilon > 0 \) such that \( \epsilon \leq \theta_0 \leq \frac{1}{\epsilon} \), \( \forall i = 1, \ldots, d \).

AML ALGORITHM:

\[
dX_t = \frac{O_t \phi_t}{\phi_t^T \theta_t} (dN_t - \phi_t^T X_t dt), X_0
\]

\[
dQ_t = -\frac{Q_t \phi_t \phi_t^T}{\phi_t^T \theta_t} dt, \quad Q_0 > 0
\]

\[
\hat{\theta}_t = I_{1u} I_{2u} X_t + \epsilon (1 - I_{1u}) \mathbf{1} + \epsilon^{-1} (1 - I_{2u}) \mathbf{1}
\]

\[
I_{1u} = 1_{\{s^*, s \to \Phi_{-1}\}}
\]

\[
I_{2u} = 1_{\{s^*, s \to \Phi_{-1}\}}
\]

COMMENT: Introducing the \( \epsilon \) above is done to establish a.s. convergence of \( \{\hat{\theta}_t\} \) to \( \theta_0 \). If we compare (2) to the AML algorithm in [6] we see that we use the extra indicator process \( I_2 \). Clearly we require knowledge of \( \epsilon \) to compute the \( \hat{\theta}_t \). The proof of \( \hat{\theta}_t \to \theta_0 \) a.s. that we will give parallels to a certain extent the procedure in [1]. First we state an auxiliary result.

Define \( \overline{Q}_t^{-1} = \int_0^t \frac{\phi_s \phi_s^T}{\phi_s^T \theta_s} ds \). Denote by \( \lambda_\text{min} \) the minimal eigenvalue of \( \overline{Q}_t^{-1} \) and by \( \lambda_\text{max} \) its maximal eigenvalue.

**Lemma 2.** There exist constants \( c \) and \( \bar{c} \) such that

i) \( \overline{q} + \epsilon^2 \lambda_\text{max}(\overline{Q}_t^{-1}) \leq \epsilon^2 \lambda_\text{max} + \bar{c} \)

ii) \( \epsilon^2 \lambda_\text{min}(\overline{Q}_t^{-1}) \geq \epsilon^2 \lambda_\text{min} + \bar{c} \)

**Proof:** Define \( \epsilon = \inf_{|x|=1} x^T \overline{Q}_0^{-1} x \) and \( \bar{c} = \sup_{|x|=1} x^T \overline{Q}_0^{-1} x \).

Since \( \epsilon \phi_0 \mathbf{1} \leq \phi_0^T \hat{\theta}_t \leq c \phi_0 \mathbf{1} \) we have for all \( x \in \mathbb{R}^d \):

\[
x^T \overline{Q}_0^{-1} x + \epsilon^2 x^T \overline{Q}_t^{-1} x \leq x^T \overline{Q}_0^{-1} x + \epsilon^2 x^T \overline{Q}_t^{-1} x.
\]
By taking infima in (3) in the right order we get (i). The second assertion follows by taking suprema.

**Theorem 3:** Consider the AML algorithm (2). Assume that $\lambda_i \to \infty$ a.s. and that there exists a function $f : [0, \infty) \to [0, \infty)$ such that $\lim_{x \to \infty} f(x) = \infty$ and such that

$$\sup_{i \geq 0} \frac{f(\log \lambda_i)}{\lambda_i} < \infty \text{ a.s.}$$

Then $\hat{\theta}_i \to \theta_0$ a.s.

**Remarks**

1. Observe that $\lambda_i \to \infty$ a.s. implies that $N_i \to \infty$ a.s. because

$$\int_0^t \phi_i \theta_0 \, ds = \theta_0 \bar{Q}_t^{-1} \theta_0 = \lambda_i \theta_0 \theta_0.$$

2. A possible choice of $f$ that can be found in the literature [1, 5] is $f(x) = x^{1-a}$, with $a > 0$.

The crucial step in the proof of theorem 3 is lemma 4 below. We will postpone the proof of this lemma and show first, after stating the lemma, how we use it in the proof of theorem 3.

**Lemma 4:** Consider (2). Let $\tilde{X}_t = X_t - \theta_0$ and $\bar{Q}_t = \bar{Q}_t^{-1} \tilde{X}_t$. Then $P_t = O(\log \lambda_t)$ a.s. ($i \to \infty$).

**Proof of theorem 3:**

$$\tilde{X}_t^T \tilde{X}_t = \tilde{X}_t^T \bar{Q}_t^\dagger \bar{Q}_t \bar{X}_t \leq \lambda_{\max}(\bar{Q}_t) P_t = \frac{P_t}{\lambda_{\max}(\bar{Q}_t^{-1})} = \frac{f(\log \lambda_t)}{\lambda_t} \frac{\lambda_t}{\lambda_{\max}(\bar{Q}_t^{-1})} \frac{\log \lambda_t}{f(\log \lambda_t) \log \lambda_t}$$

Consider the right hand side of (4). Its last factor is bounded in view of lemma 4. The first factor is bounded because of the assumption in the theorem. The second factor is bounded because of lemma 2 and the third factor tends to zero because of the assumption on $f$. We conclude that $\hat{\theta}_i \to \theta_0$ a.s. But now it is easy to show that $\theta_i \to \theta_0$ a.s.

$$\tilde{\theta}_i = \hat{\theta}_i - \theta_0 = \tilde{X}_t^T P_t (1 - I_{2}) (e^{1} \theta_0) + (1 - I_{2})(e^{\eta} - 1).$$

Since $\phi_t^T \theta_0 > \phi_t^T \epsilon$ there is $\eta > 0$ such that $\phi_t^T \theta_0 \geq \phi_t^T (e^{\eta} \epsilon + \eta)$. Because $\tilde{X}_t \to 0$ we eventually have $|\tilde{X}_t| < \eta, \forall i$. But then

$$\phi_t^T \tilde{X}_t = \phi_t^T \tilde{X}_t + \phi_t^T \epsilon \theta_0 \geq - \phi_t^T \epsilon \eta + \phi_t^T \epsilon (e^{\eta} \epsilon + \eta) = \phi_t^T \epsilon (e^{\eta} \epsilon + \eta).$$

Therefore $I_{1} \to 1$. In a similar way one can prove that $I_{2} \to 1$, which implies...
that \( \tilde{\theta}_t \to 0 \) a.s. \( \Box \).

The proof of lemma 4 involves a series of other lemmas.

**Lemma 5:** Let \( P_0 > 0, P_0 \in \mathbb{R}^{k \times k} \) and let \( P_t := P_0 + \int_0^t \xi(s)^T \xi(s) ds \) for a left continuous function \( \xi: [0, \infty) \to \mathbb{R}^k \). Then

(i) \( \int_0^t \xi(s)^T P_s^{-1} \xi(s) ds = \log \det(P_s) - \log \det(P_0) \)

(ii) \( \int_0^t \xi(s)^T P_s^{-1} \xi(s) ds = O(\log \lambda_{\text{max}}(P_t)) \).

**Proof:** [1].

**Lemma 6:** Let \( m \) be a quasi left-continuous locally square integrable martingale with \( <m> \equiv A \). Let \( f: [0, \infty) \to [0, \infty) \) be a differentiable increasing function with

\[ \lim_{x \to \infty} f(x) = \infty \text{ and } \int_0^\infty \frac{dx}{(1 + f(x))^2} < \infty. \]

Define \( g_t := 1 + f(A_t) \). Then both \( g_t^{-1} m_t \) and \( g_t^{-2}[m,m]_t \) converge almost surely for \( t \to \infty \). On \( \{ A_\infty = \infty \} \) both limits equal zero a.s.

**Proof:** This is a simple application of lemma A. Consider \( g_t^{-1} m_t \). Define \( X_t = g_t^{-2} m_t^2 \). Then application of the stochastic calculus rule yields

\[
dX_t = -2g_t^{-3} f(A_t) m_t^2 dA_t + g_t^{-2}(2m_t dm_t + d[m,m]_t)
\]

\[
= -2g_t^{-3} f(A_t) X_t dA_t + g_t^{-2} dA_t + g_t^{-2}(2m_t dm_t + d[m,m]_t - A_t))
\]

Notice that \( f(A_t) > 0 \). Application of lemma A immediately yields the desired result since

\[
\int_0^\infty g_t^{-2} dA_t = \int_0^\infty \frac{dx}{(1 + f(x))^2} < \infty.
\]

On \( \{ A_\infty = \infty \} \) lemma A also yields that \( X_t \to 0 \) because

\[
\int_0^\infty g_t^{-1} f(A_t) X_t dA_t = \int_0^\infty X_t d\log g_t.
\]

The statement about \( g_t^{-2}[m,m]_t \) can be proved similarly. \( \Box \).

**Remarks**

1. The statements of the lemma can be summarized as

\( m_t = o(g_t) + O(1) \) and \( [m,m]_t = o(g_t^2) + O(1) \).

2. Of course we may replace \( g_t \) in the lemma by \( f(A_t) \) since we consider the behaviour for \( t \to \infty \).

3. Convenient choices of \( f \) in applications are \( f(x) = x^{\alpha(1+a)} \), with \( a > 0 \).
Proof of Lemma 4: For \( \tilde{X} \) we have the following equation

\[
d\tilde{X}_t = \frac{Q_t \phi_t}{\phi_t^T \theta_t} (d\mu_t - \phi_t^T \tilde{X}_t dt)
\]

Hence

\[
dP_t = d(\tilde{X}_t^T \theta_t^{-1} \tilde{X}_t) = 2 \frac{\phi_t^T \tilde{X}_t}{\phi_t^T \theta_t} (d\mu_t - \phi_t^T \tilde{X}_t dt) + \frac{\phi_t^T Q_t \phi_t}{\phi_t^T \theta_t^2} dt + \frac{\phi_t^T Q_t \phi_t}{(\phi_t^T \theta_t^-)^2} dN_t
\]

or

\[
P_t - P_0 + \int_0^t \frac{(\tilde{X}_s^T \phi_s)^2}{\phi_s^T \theta_s} ds =
\]

\[
2 \int_0^t \frac{\phi_s^T \tilde{X}_s}{\phi_s^T \theta_s} d\mu_s + \int_0^t \frac{\phi_s^T Q_s \phi_s}{(\phi_s^T \theta_s)^2} \phi_s^T \theta_s ds + \int_0^t \frac{(\phi_s^T Q_s \phi_s)^2}{(\phi_s^T \theta_s^-)^2} dN_s
\]

(6)

Write (5) in obvious notation as

\[
\sum_{i=1}^r L_t = 2M_{1r} + R_t + M_{2r}
\]

(7)

Compute

\[
\left<M_1\right>_t = \int_0^t \frac{\phi_s^T \tilde{X}_s}{\phi_s^T \theta_s} \phi_s^T \theta_s ds.
\]

Observe that

\[
\epsilon^2 \leq \frac{\phi_s^T \theta_s}{\phi_s^T \theta_s} \leq \epsilon^2
\]

Hence \( \epsilon^2 L_t \leq \left<M_1\right>_t \leq \epsilon^{-2} L_t \). Hence \( M_r = o(L_t) + O(1) \) in view of lemma 6 (take \( f(x) = x \), and remarks 1 and 2 that follow this lemma. Consider now \( R_t \) and notice that

\[
\epsilon^2 \int_0^t \frac{\phi_s^T Q_s \phi_s}{\phi_s^T \theta_s} ds \leq R_t \leq \epsilon^{-2} \int_0^t \frac{\phi_s^T Q_s \phi_s}{\phi_s^T \theta_s} ds
\]

(8)

The integrals in the extreme sides of (8) are of the form encountered in lemma 5. (Take \( \xi_t = \frac{\phi_t}{(\phi_t^T \theta_t)^2} \)). Therefore \( R_t = O(\log \lambda_{\max}(Q_t^{-1})) \). The last term to analyze in (7) is \( M_{2r} \).

\[
\left<M_2\right>_t = \int_0^t \frac{\phi_s^T Q_s \phi_s}{(\phi_s^T \theta_s)^2} \phi_s^T \theta_s ds = \int_0^t \frac{\phi_s^T Q_s \phi_s}{(\phi_s^T \theta_s)^2} \phi_s^T \theta_s ds
\]

\[
\leq \int_0^t \frac{\phi_s^T \phi_s}{\phi_s^T \theta_s} ds \leq \epsilon^{-1} \int_0^t d\text{tr}(Q_s) \leq \epsilon^{-4} \int_0^t \text{tr}(-Q_s) ds
\]

\[
\leq \epsilon^{-4} \text{tr}(Q_0) < \infty.
\]
From lemma 6 we conclude that \( \frac{M_2}{<M_2>} \) converges to a finite limit and since \( <M_2>, <\epsilon^{-4}tr(Q_0) > \) \( M_2 \) is a.s. bounded. Collecting the above results we get from (4.37)

\[
P_t - P_0 + L_t = o(L_t) + O(1) + O(\log \lambda_{\text{max}}(Q_t^{-1})) + O(1)
\]
or

\[
P_t - P_0 + L_t (1 + o(1)) = O(1) + O(\log \lambda_{\text{max}}(Q_t^{-1}))
\]

From lemma 2 we obtain after dividing by \( \log \lambda_t \)

\[
\frac{P_t}{\log \lambda_t} + (1 + o(1)) \frac{L_t}{\log \lambda_t} = O(1)
\]

Since both \( P_t \) and \( (1 + o(1))L_t \) are (eventually) nonnegative we get \( P_t = O(\log \lambda_t) \), as was to be proven. \( \square \).

We close this section by proving that the limit distribution of the AML estimators defined by (2) is asymptotically normal.
Theorem 7: Assume that \( \{\delta_t\} \) given by (2) is a.s. convergent. Assume that there exist \( P: [0,\infty) \to \mathbb{R}^{d \times d} \) and \( h: [0,1] \times [0,\infty) \to [0,\infty) \) such that

1. \( h \) is increasing in each of its arguments and for all \( t,T \)
   
   \[ h(t,T) \leq h(1,T) - T. \]

2. all \( P(t) \) are symmetric positive definite for \( t > 0 \), \( P \) is increasing to infinity, continuous, and
   
   \[ R(t) = \lim_{T \to \infty} P(T)^{-\frac{1}{2}} P(h(t,T)P(T)^{-\frac{1}{2}} \exists \text{ and is positive definite for all } t > 0. \]

3. \( P(t)^{\frac{1}{2}} Q_{\delta_t} \to I \) in probability.

Then \( Q_{\delta_t} \overset{d}{\to} N(0,I) \).

PROOF: Since both \( I_{1,t} \) and \( I_{2,t} \) tend to 1, eventually \( X_t = \delta_t \). Therefore it is sufficient to prove that \( Q_{\delta_t} \overset{d}{\to} N(0,I) \).

From (2) we obtain \( Q_{\delta_t} \overset{d}{\to} N(0,I) \), where

\[
M_t = \int_0^t \frac{\varphi - \varphi_0}{0 \varphi^T \varphi} dm.
\]

Hence the asymptotic law of \( Q_{\delta_t} \overset{d}{\to} N(0,I) \) is the same as that of

\[
\hat{Q}_{\delta_t} \overset{d}{\to} \hat{Q}_{\delta_t} \overset{d}{\to} M_t. \tag{9}
\]

Like in [6] it is easy to prove that

\[
\hat{Q}_{\delta_t} \overset{d}{\to} Q_0 \overset{d}{\to} Q_{\delta_t} \overset{d}{\to} I \quad \text{a.s.} \tag{10}
\]

Hence it suffices to establish that the asymptotic law of \( \hat{Q}_{\delta_t} M_t \) is \( N(0,I) \).

Now \( \left< M_t \right>_{t} = \int_0^t \frac{\varphi_s \varphi^T_s}{\varphi^T_s \varphi_s} \left[ \varphi^T_s \varphi_s \right] ds. \)

In a similar way as proving (10), it is possible to show that

\[
\hat{Q}_{\delta_t} \overset{d}{\to} \left< M_t \right>_{t} \overset{d}{\to} I \quad \text{a.s.}
\]

Then using condition (iii) of the theorem, it follows that
Define now for each $T \in (0, \infty)$ and $\lambda \in \mathbb{R}^d$ a new martingale (w.r.t. the filtration $(F_{h(t,T)})_{t \in (0,T)}$) $Z^T, \lambda$ by $Z^T, \lambda = \lambda^T P(T)^{-\frac{1}{2}} M_{h(t,T)}$. 

Let now $W$ be some continuous Gaussian martingale with quadratic variation $<W>_t = \lambda^T R(t) \lambda$. Such a $W$ exists on a suitable filtered probability space, since $R(t)$ is continuous increasing.

We claim

$$Z^T, \lambda \not\leq W.$$  \hspace{1cm} (12)

We prove (12) by checking the conditions of lemma B. Compute

$$<Z^T, \lambda>_t = \lambda^T P(T)^{-\frac{1}{2}} <M>_{h(t,T)} P(T)^{-\frac{1}{2}} \lambda =$$

$$= \lambda^T P(T)^{-\frac{1}{2}} P(h(t,T))^{-\frac{1}{2}} (P(h(t,T))^{-\frac{1}{2}} <M>_{h(t,T)} P(h(t,T))^{-\frac{1}{2}}) P(h(t,T))^{\frac{1}{2}} P(T)^{-\frac{1}{2}} \lambda.$$

From condition (ii) and (11) we then obtain for $T \to \infty$

$$<Z^T, \lambda>_t \to \lambda^T R(t) \lambda$$

which corresponds to condition (i) of lemma B.

Observe that for the jumps of $Z^T, \lambda$ we have

$$|\Delta Z^T, \lambda| = \left| \lambda^T P(T)^{-\frac{1}{2}} \frac{\phi_t^T}{\phi_t} \frac{\phi_t}{\theta_t} \right| \leq \lambda^T P(T)^{-\frac{1}{2}} \lambda \leq \lambda^T P(T)^{-1} \lambda \leq \epsilon^{-2} \lambda^T P(T)^{-1} \lambda.$$

Hence the jumps of $Z^T, \lambda$ are bounded by a deterministic quantity that tends to zero. Hence also the second condition of lemma B is satisfied and (12) follows. In particular $\lambda^T P(T)^{-\frac{1}{2}} M_t = Z^T, \lambda \not\leq N(0, \lambda^T \lambda)$. Finally by noticing that $Q^T M_t = Q^T P(T)^{-\frac{1}{2}} P(t)^{\frac{1}{2}} M_t$ and by using condition (iii) again, we have finished the proof.

As a final remark we mention that the behaviour of this AML algorithms in general will be superior to a least squares algorithm like in [1].
The easiest way, although it does not give a complete account, for this to see, is to assume that the process \( \varphi \) in (1) is deterministic. Then the Fisher information matrix at time \( t \) becomes \( \mathcal{Q}_t^{-1} \). Hence from theorem 7 we see that our estimators have an asymptotic variance that equals the Cramer-Rao bound. It is also this observation that led us to considering the algorithm (2).

APPENDIX

The next lemma generalizes a result in [7].

Lemma A: Let \( X \) be a nonnegative stochastic process such that \( X_t - X_0 + A_t - B_t + M_t \). Here \( A \) and \( B \) are predictable increasing processes with \( A_0 = B_0 = 0 \) and \( M \) is a local martingale. Assume that \( \lim_{t \to a} A_t < \infty \) a.s. Then both \( \lim_{t \to a} X_t \) and \( \lim_{t \to a} B_t \) exist and are finite.

PROOF: Without loss of generality we assume that \( X_0 = 0 \) a.s. Let \( (T_n) \) be a fundamental sequence for \( M \) [3]. Let \( (S_n) \) be stopping times defined by \( S_n = \inf \{ t > 0 : A_t > n \} \). Each \( S_n \) is then predictable and hence there exist for each \( n \) another sequence of stopping times \( (S_n', k > 0) \) announcing \( S_n \). Define \( R_n = \sup \{ S_n', k : k < n \} \). Then \( R_n < S_n \) and \( A_{R_n} < n \). Furthermore \( (R_n \to \infty) \). Now for all \( k, n \) \( \{ M_{t \wedge T_n} \wedge R_k \}_{t \geq 0} \) is a uniformly integrable martingale, and

\[
M_{t \wedge T_n} \wedge R_k \geq X_{t \wedge T_n} \wedge R_k - A_{t \wedge T_n} \wedge R_k \geq -k.
\]

In particular \( \{ M_{t \wedge T_n} \wedge R_k \}_{n \geq 0} \) is uniformly integrable \( (M_s - \max(0, M_s)) \). Hence

\[
E[M_{t \wedge R_k} | F_s] - E[\lim_{n \to \infty} M_{t \wedge T_n} \wedge R_k | F_s] \leq \liminf_{n \to \infty} E[M_{t \wedge R_k} | T_n | F_s] - \liminf_{n \to \infty} M_{g \wedge R_k} \wedge T_n = M_{g \wedge R_k}.
\]

Here the inequality follows from Fatou's lemma. So we see that \( \{ M_{t \wedge R_k} \}_{t \geq 0} \) is a supermartingale with \( M_{t \wedge R_k} \leq k \). Hence the convergence theorem [3] for supermartingales tells us that
lim_{t\to\infty} M_{\text{CAR}_k} exists and is finite a.s. But then also
\lim_{t\to\infty} (X_{\text{CAR}_k} + B_{\text{CAR}_k}) exists and is finite. Since both $X_{\text{CAR}_k}$ and
$B_{\text{CAR}_k}$ are nonnegative and $B$ is increasing, we obtain that both
\lim_{t\to\infty} X_{\text{CAR}_k}$ and \lim_{t\to\infty} B_{\text{CAR}_k}$ exist and are finite.

On the set $\{R_x = \infty\}$ these limits equal \lim_{t\to\infty} X_t$ and \lim_{t\to\infty} B_t$ respectively. But $(R_x = \infty) \subset \Omega$, which finishes the proof. \qed

The following lemma is a special case of a much more general result on
weak convergence of locally square integrable martingales, that can be
found in for instance the monograph by Jacod & Shiryaev [4].

Lemma B (Central limit theorem). Let $M,M^n, n \geq 0$ be real valued locally
square integrable martingales defined on a suitable filtered probabil-
ity space. Let $M$ be a continuous Gaussian martingale with $C_t = \langle M_t \rangle = EM_t^2$. Assume that the following two conditions hold:

(i) $\langle M^n \rangle_t \to C_t$ in probability as $n \to \infty$ for all $t$.

(ii) $\sup_t |\Delta M^n_t| \leq c_n$, where $(c_n)$ is a deterministic sequence with
$\lim_{n \to \infty} c_n = 0$.

Then $(M^n)$ converges weakly to $M$ for $n \to \infty$. Notation: $M^n \overset{w}{\to} M$.

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<thead>
<tr>
<th>Title</th>
<th>Authors</th>
</tr>
</thead>
<tbody>
<tr>
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<td>H. Visser</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>M.M. van Dijk</td>
</tr>
<tr>
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<td>M.M. van Dijk, M. Ruuswijk</td>
</tr>
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<td>H. Linnebar, C.P. van Beers</td>
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<td>N.M. van Dijk</td>
</tr>
<tr>
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<td>J.C.H. van Ommeren</td>
</tr>
<tr>
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<td>J.C.H. Tijms</td>
</tr>
<tr>
<td>Export Agriculture and Labour Market in Nicaragua</td>
<td>J.P. de Groot, H. Ciemens</td>
</tr>
<tr>
<td>Patterns of South-South trade in manufactures</td>
<td>H. Verbruggen, J. Huisitus</td>
</tr>
<tr>
<td>Asymptotic analysis for buffer behaviour in communication systems</td>
<td>H. C. Tijms, J.C.H. van Ommeren</td>
</tr>
<tr>
<td>A non-exponential queueing system with batch servicing</td>
<td>N.M. van Dijk, E. Smetting</td>
</tr>
<tr>
<td>Existence and uniqueness of stochastic price equilibria in heterogeneous markets</td>
<td>J. Rouwendal</td>
</tr>
<tr>
<td>SQTP, the structure of protection and South-South trade in manufactures</td>
<td>H. Verbruggen</td>
</tr>
<tr>
<td>Female participation in agriculture in the Dominican Republic</td>
<td>N. van Heiland, M. R. Herreweijer, J. de Groot</td>
</tr>
<tr>
<td>Product Forms for Random Access Schemes</td>
<td>N.M. van Dijk</td>
</tr>
<tr>
<td>Adaptive Forecasting with Hyperfilters</td>
<td>A.H.G.N. Merkies, J. Rouwendal</td>
</tr>
<tr>
<td>Specification and Estimation of a Logit Model for Housing Choice in the Netherlands</td>
<td>J. Rouwendal</td>
</tr>
<tr>
<td>An elementary proof of a basic result for the GI/G/1 queue</td>
<td>J.C.H. van Ommeren, R.C. Nobel</td>
</tr>
<tr>
<td>A Note on Consistent Estimation of Heteroskedastic and Autoregressive Covariance Matrices</td>
<td>H. Kool</td>
</tr>
<tr>
<td>Risk Aversion and the Family Farm</td>
<td>C.P.J. Burger</td>
</tr>
<tr>
<td>Networks with mixed-processor sharing parallel queues and common pools</td>
<td>N. van Dijk, J.F. Akyildiz</td>
</tr>
<tr>
<td>Technogenesis: Incubation and Diffusion</td>
<td>D.J.F. Kamann, P. Nijkamp</td>
</tr>
<tr>
<td>A Household Life Cycle Model for the Housing Market</td>
<td>P. Nijkamp, L. van Wissen, R. Rima</td>
</tr>
<tr>
<td>Qualitative Impact Analysis For Dynamic Spatial Systems</td>
<td>P. Nijkamp, M. Sonis</td>
</tr>
<tr>
<td>Interactive MultiCriteria Decision Support For Environmental Management</td>
<td>R. Janssen, P. Nijkamp</td>
</tr>
<tr>
<td>Stochastic Market Equilibria with Rating and Limited Price Flexibility</td>
<td>J. Rouwendal</td>
</tr>
<tr>
<td>Theory of Chaos in a Space-Time Perspective</td>
<td>P. Nijkamp, A. Meggiani</td>
</tr>
<tr>
<td>R &amp; D Policy in Space and Time</td>
<td>P. Nijkamp, J. Poot, J. Rouwendal</td>
</tr>
<tr>
<td>Dynamics in Land Use Patterns Socio-Economic and Environmental Aspects of the Second Agricultural Land Use Revolution</td>
<td>P. Nijkamp, F. Soeteman</td>
</tr>
<tr>
<td>Endogenous Production of R &amp; D and Stable Economic Development</td>
<td>J. Rouwendal, P. Nijkamp</td>
</tr>
<tr>
<td>Multicriteria Methods: Een gevaligheidsanalyse aan de hand van de vastigingsproblematiek van kerncentrales</td>
<td>J.A. Hartog, E. Hinlopen, P. Nijkamp</td>
</tr>
<tr>
<td>The Development Potential of High Tech Firms in Backward Areas - A Case study for the Northern Part of The Netherlands</td>
<td>R. van der Mark, P. Nijkamp</td>
</tr>
<tr>
<td>The Duration of Unemployment: Stocks and Flows on Regional Labour Markets in the Netherlands</td>
<td>C. Gorter, P. Nijkamp, P. Rietveld</td>
</tr>
<tr>
<td>Parametrization of simplicial algorithms with an application to an empirical general equilibrium model</td>
<td>M. Hofkes</td>
</tr>
</tbody>
</table>