PRACTICAL APPROXIMATIONS FOR FINITE-BUFFER QUEUEING MODELS WITH BATCH-ARRIVALS

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Practical approximations for finite-buffer queueing models with batch-arrivals.

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ABSTRACT

In this paper we discuss the finite capacity queueing model $G^b/G/1/N$ with batch-arrivals, a single server and having room for only $N$ customers. For this model two different rejection strategies are conceivable: a batch finding upon arrival not enough space in the buffer is rejected completely or the buffer is filled up and only a part of the batch is rejected. For either strategy we are interested in the rejection-probabilities both for a batch and for an individual customer. Also we want to investigate the waiting-time distribution for an accepted customer. In general we cannot find analytical solutions for this model. However by specifying the service-time distribution to be an Erlang-$k$ distribution, a Markov-chain approach is possible and exact results can be obtained. The next step is to get approximate results for the general case via interpolation with respect to the squared coefficient of variation of the service-time. We give approximations for the waiting-time percentiles and for the minimal bufferspace such that the rejection-probability is below a prespecified level. Also numerical results are given to illustrate the quality of the approximations.
0. INTRODUCTION

In modern telecommunication-technology and computer networks we are confronted with the phenomenon that messages are sent in batches over a communication-line and must be buffered at their destination-node before they can be handled. The messages are handled one at a time in order of arrival and within a batch in random order. Batches which upon arrival find not enough space in the buffer are, at least partly, lost. So a natural design problem arises with respect to the buffersize required to assure a reasonable behaviour of the system, given the probability-distribution of the batchsize, the interarrival-time of the batches and the service-time of a message. The service level can be measured in terms of an upperbound for the rejection-probability of a message or batch, or in terms of the waiting-probability for an accepted message. This design problem motivates the study of the $G^X/G/1/N$ model for which we present approximative results using the exact solutions of the $G^X/E_k/1/N$ model, where $E_k$ stands for an Erlang-$k$ distributed service-time. In this special case we can interpret the service-time as the sum of $k$ independently identically distributed exponential phases. Thus the total number of uncompleted phases seen by an arriving batch forms a sufficient state-description to enable an embedded Markov-chain approach.

A simpler analysis is possible in case of Poisson arrivals ($M^X/E_k/1/N$) because under this specific arrival process the total number of uncompleted phases at an arbitrary epoch forms a continuous Markov-chain. Then we can find a recursive solution for the steady-state probabilities. In section 1 we shall discuss the $M^X/E_k/1/N$ model and derive the formulae for the rejection-probabilities for a batch and an individual customer. Also the waiting-time distribution for an accepted customer will be given.

In section 2 we give the exact solution for the above mentioned $G^X/E_k/1/N$ model. In section 3 we discuss the general $G^X/G/1/N$ case and use the exact results for the special case of the $G^X/E_k/1/N$ model to get approximative results for both the waiting-time percentiles and the minimal buffersize for which the rejection-probability does not exceed a given value. The approximations involve interpolation with respect to the squared coefficient of variation of the service-time.
In section 4 at last we give some comparisons between approximate values and exact values for models which also allow an analytical approach.

1. The $M^X/E_1/N$ queueing model

For the $M^X/E_1/N$ model we can give a recursive solution for the steady-state probabilities of the total number of uncompleted phases at an arbitrary epoch via a continuous Markov-chain analysis. We describe the model and this analysis in subsection 1.1. Once these probabilities are known we can derive the rejection-probabilities for a batch and an individual customer for each of the following two rejection/acceptance strategies. If an arriving batch does not fit completely in the remaining capacity of the buffer, then

a. under the whole batch acceptance strategy (WBAS) the whole batch is rejected.

b. under the partial batch acceptance strategy (PBAS) the remaining places in the buffer are filled up and only the customers for whom there is no place left are rejected.

Note that under the PBAS strategy every customer of a partially rejected batch has the same probability to be rejected because of the random order of customers within a batch. Under the WBAS strategy this probability equals one independent of the service order.

In the second subsection we discuss the waiting-time distribution of an accepted customer.

1.1. Description of the model and its Markov-chain analysis.

Batches of customers arrive at a service-facility according to a Poisson process with rate $\lambda$. The batch size $X$ has a general discrete probability distribution: $P(X = k) = \alpha_k (k = 1, 2, \ldots)$. The service-time $S$ for an individual customer has an Erlang-$r$ distribution with parameter $\mu$, so $E(S) = r/\mu$. The service-facility can handle one customer at a time and works at unity rate. The batch size distribution is independent of the arrival process and the service-times.

For customers who cannot be taken into service immediately there is a
buffer of \( N - 1 \) waiting-places. As soon as the server becomes free a new customer is taken into service. Here we assume a first-come-first-served (FCFS) queue discipline for customers from different batches and a random selection for service (RSS) queue discipline for customers from the same batch. If no new customer is present the server becomes idle.

If an arriving batch contains too many customers to fit into the remaining places of the buffer we make a distinction between the two already mentioned rejection/acceptance strategies: WBAS and PBAS. So we have to deal with two different models, which we will discuss separately. Using that the Erlang-\( k \) distributed service-time can be interpreted as a sum of \( k \) identically, independently exponentially distributed phases, we can describe for both strategies the state of the system at an arbitrary epoch \( t \) as

\[
X(t) = \text{the total number of uncompleted phases present in the system at time } t. 
\]

We now have that \( (X(t), t \geq 0) \) is a continuous Markov-chain, where the steady-state probabilities can be calculated via the standard technique of equating the rate at which the process \( (X(t)) \) leaves any state to the rate at which the process enters that state. The state-space of \( (X(t)) \) is \( \{0, 1, \ldots, N_r\} \).

Let \( f_i \) denote the time-average probability of having \( i \) uncompleted phases in the system at an arbitrary time i.e.

\[
f_i = \lim_{t \to \infty} P(X(t) = i) 
\]

By the property 'Poisson arrivals see time-averages' (PASTA) we have that \( f_i \) can also be interpreted as the probability that a batch sees \( i \) uncompleted phases in the system upon its arrival.

Next we define

\[
p_j = \lim_{t \to \infty} P(\text{at epoch } t \text{ there are } j \text{ customers in the system}).
\]

So \( (p_j) \) is the time-average distribution for the number of customers in the system at an arbitrary time. Clearly, we have

\[
p_j = \sum_{k=(j-1)r+1}^{jr} f_k 
\]

By the PASTA property \( p_j \) is also the probability that a batch finds upon arrival \( j \) other customers already present in the system.

Our first task is now to give the balance equations and the rejection-
probabilities for the two different strategies WBAS and PBAS.

A recursive equation for the state probabilities is obtained by applying the balance principle,

\[ \text{the rate at which the process leaves a set } A \text{ of states} = \text{the rate at which the process enters that set of states} \]

to an appropriately chosen set \( A \) of states rather than to a single state.

If we take \( A = \{j, \ldots, Nr\} \) \((j = 1, 2, \ldots)\) we get in case of a WBAS strategy the following equations:

\[
\mu * f_j = \lambda * \sum_{i=0}^{j-1} f_i * \sum_{k>(j-1)/r} \sigma_k \cdot \left\lfloor \frac{(N-i)}{r} \right\rfloor
\]

Here \( \left\lfloor x \right\rfloor \) stands for the integer part of \( x \). Under a PBAS strategy the balance equations show only a minor difference:

\[
\mu * f_j = \lambda \sum_{i=0}^{j-1} f_i * \sum_{k>(j-1)/r} \sigma_k
\]

In either case we can solve the given system of equations recursively by starting with \( f_0 = 1 \). The steady-state probabilities \( f_j \) can subsequently be found by using the normalizing equation

\[
\sum_{j=0}^{Nr} f_j = 1
\]

We now consider the rejection-probabilities under the WBAS strategy.

First we look at the probability that a batch is rejected. This can be done easily by conditioning on the number of customers present upon arrival of the batch:

\[
P(\text{batch rejected}) = \sum_{k=0}^{N} P(\text{batch rejected} | \text{batch sees } k \text{ customers upon arrival}) * p_k = \sum_{k=0}^{N} \frac{N}{k} P(\text{batch consists of more than } N-k \text{ customers}) * p_k
\]

\[
= \sum_{k=0}^{N} P_k * \left[ 1 - \sum_{i=1}^{N-k} \alpha_i \right]
\]

For the calculation of the probability that a customer is rejected we need the
following renewal-theoretic result [BURKE]:

\[ P(\text{an arbitrary customer belongs to a batch of size } i) = i \alpha_i / E(X). \]  

(2)

Now we can deduce

\[
P(\text{customer rejected}) = \sum_{k=0}^{N} P(\text{customer sees } k \text{ customers rejected already present upon arrival}) \cdot p_k
\]

\[
- \sum_{k=0}^{N} P(\text{customer belongs to a batch of size more than } N-k) \cdot p_k
\]

\[
- \sum_{k=0}^{N} p_k \left( 1 - \sum_{i=1}^{N-k} \alpha_i / E(X) \right).
\]

(3)

Next we look at the rejection-probabilities under PBAS. Under this strategy we prefer to speak of 'batch overflow' when an arriving batch does not find space enough for all its customers, because under PBAS we have only partial batch-rejection. Apart from this change in name the formula for the probability of batch-overflow is exactly the same as formula (1) for the probability of batch-rejection under WBAS, where of course the probabilities \( p_k \) are different. Then

\[
P(\text{batch overflow}) = \sum_{k=0}^{N} p_k \left( 1 - \sum_{i=1}^{N-k} \alpha_i \right)
\]

To find the rejection-probability for an individual customer we define:

\( \eta_j = P(\text{an arbitrary customer takes the } j\text{-th position in his batch}). \)  

(4)

From the formula (2) we get:

\[
P(\text{an arbitrary customer takes the } j\text{-th position in his batch}) = \sum_{i=j}^{\infty} \alpha_i / E(X), \ j \geq 1
\]

Now we can deduce

\[
P(\text{customer rejected}) = \sum_{k=0}^{N} P(\text{customer rejected | which sees } k \text{ customers already present upon arrival}) \cdot p_k
\]

\[
- \sum_{k=0}^{N} P(\text{customer takes a position greater than } N-k \text{ in his batch}) \cdot p_k
\]

\[
- \sum_{k=0}^{N} p_k \left( 1 - \sum_{i=1}^{N-k} \eta_i \right).
\]

(5)
1.2. The waiting-time distribution for an accepted customer

Let $W_q$ be defined as the waiting-time of an accepted customer. To find the probability distribution of $W_q$, we define the random variable

$F =$ the total number of uncompleted phases in front of an arbitrarily accepted customer just after his entrance into the system.

Using the fact that the waiting-time of a customer having $j$ phases in front of him has an Erlang-$j$ distribution, we have for either strategy:

$$P(W_q > x) = \sum_{j=1}^{(N-1)r} P(W_q > x \mid F=j) \cdot P(F=j) = \sum_{j=1}^{(N-1)r} \sum_{i=0}^{j-1} \frac{(\mu X)^i}{i!} \frac{1}{P(F=j)} \cdot \frac{1}{P(F=j)} \cdot \frac{1}{e^{\mu X} (\mu X)^i / i!}$$

So it remains to calculate the distribution of $F$ for either of the two strategies. To compute the probability distribution of $F$, let us define the following events:

$A =$ the event that an arbitrary customer is accepted

$A_j =$ the event that an arbitrary customer is accepted and has $j$ uncompleted phases in front of him just after his entrance into the system.

Then, by the definition of conditional probability,

$$P(F=j) = P(A_j) / P(A).$$

For the two strategies WBAS and PBAS the rejection probability $1-P(A)$ is given by the respective formulae (3) and (5). To find $P(A_j)$, let $B_{ki}$ denote the joint event that an arbitrary customer belongs to a batch of size $k$ and that $i$ uncompleted phases are in the system just prior to the arrival of his batch. Using (2) we have,

$$P(B_{ki}) = \frac{(k\alpha / E(X)) \cdot f_i}{(N-1)r}.$$ 

By the law of total probability, we have for any fixed $j$

$$P(A_j) = \sum_{k,i} P(A_j \mid B_{ki}) \cdot P(B_{ki}).$$

Noting that each newly arriving customer represents exactly $r$ phases, it follows that for fixed $j$ the probability $P(A_j \mid B_{ki})$ cannot be positive unless $i$ and $k$ satisfy $i \leq j$ and $j-i \in (0, r, \ldots, kr)$ and $k \leq u_i$, where

$$u_i = \left\{ \begin{array}{ll} \frac{(N-1)}{r} & \text{for the WBAS strategy} \\ \infty & \text{for the PBAS strategy} \end{array} \right.$$ 

In case $i$ and $k$ satisfy the above characteristics we have $P(A_j \mid B_{ki}) = 1/k$,
otherwise \( P\{A_j | B_{k1}\} = 0 \). Together the formulae (8) - (10) yield for both strategies the result

\[
P\{A_j\} = \sum_{i=0}^{\infty} \sum_{k=\lceil (j-i)/r \rceil +1}^{\infty} \frac{u_i}{k^2} \frac{P\{A_j | B_{k1}\} * f_1 * \alpha_k / E(X)}{E(X)}.
\]

(11)

Substituting (11) in (7) gives the desired probability \( P\{F = j\} \).

2. The \( G^x/E_x/1/N \) queueing model

For the \( G^x/E_x/1/N \) model we can also give a Markov-chain analysis if we look only at the arrival epochs. This so called embedded Markov-chain approach leads to a system of equilibrium equations for the steady-state probabilities that an arriving batch sees \( j \) uncompleted phases in the system (\( j = 0, 1, .. \)). Once these probabilities are known we can use the same formulae as deduced in section 1 for the rejection-probabilities and the waiting-time distribution. Thus we can confine ourselves to the description of the model and its Markov-chain analysis. This will be done in subsection 2.1, where we will also give the results for some specific interarrival-time distributions.

2.1. The description of the model and its Markov-chain approach

The only difference with the model discussed in section 1 lies in the arrival process: here we consider a generally distributed interarrival-time with density-function \( g(t) \). For the other characteristics of the model we refer to the previous section.

Now define

\[
X_n = \text{the total number of uncompleted phases in the system just prior to the arrival of the n-th batch.}
\]

Then \( (X_n, n = 1, 2, ..) \) is a discrete aperiodic Markov-chain with a finite state-space \( \{0, 1, ..., Nr\} \). Let

\[
z_i = \lim_{n \to \infty} P(X_n = i) \quad (i = 0, 1, ..)
\]

be the steady-state probability that an arriving batch sees \( i \) uncompleted phases in the system. Then we know that the \( z_i \) can be determined as the
unique solution of the linear equations

\[ z_j = \sum_{i=0}^{N_r} z_i \cdot p_{ij}, \quad j = 0, 1, \ldots \quad (12) \]

together with the normalization equation

\[ \sum_{j=0}^{N_r} z_j = 1. \]

Here the \( p_{ij} \) stand for the one-step transition probabilities of the Markov-chain \( (X_n, n = 1, 2, \ldots) \). So it remains to calculate the \( p_{ij} = P(X_{n+1} = j \mid X_n = i) \).

It suffices to do this for \( j = 0 \), since

\[ p_{i0} = 1 - \sum_{j=1}^{N_r} p_{ij}. \]

Let \( D_n \) be the size of the \( n \)-th batch and let \( T_n \) stand for the interarrival-time between the \( (n-1) \)-th and the \( n \)-th batch. Then by conditioning on the number of customers in the \( n \)-th batch and the interarrival-time we get the following formula,

\[ p_{ij} = \sum_{k=0}^{\infty} \int_0^\infty P(X_{n+1} = j \mid X_n = i, D_n = k, T_{n+1} = t) \cdot g(t) \, dt \cdot \alpha_k. \quad (13) \]

Next we must calculate the conditional probability

\[ P(X_{n+1} = j \mid X_n = i, D_n = k, T_{n+1} = t). \]

To do this put for abbreviation

\[ POIS(t, h) = e^{-\mu t} \cdot (\mu t)^h / h!. \]

Note that \( POIS(t, h) \) represents the probability of \( h \) service-phase completions during a time interval of length \( t \) when the server would be continuously busy during this time. Using this observation it is not difficult to see the following results:

(i) WBAS strategy. Then for \( j \neq 0, \)

\[
\begin{align*}
P(X_{n+1} = j \mid X_n = i, D_n = k, T_{n+1} = t) &= \begin{cases} 
POIS(t, i+kr-j) & \text{if } \max((j-i)/r, 1) \leq k \leq [(N_r-i)/r] \\
POIS(t, i-j) & \text{if } i \geq j \text{ and } k > [(N_r-i)/r] \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]
(ii) PBAS strategy. Then for \( j \neq 0 \),

\[
P(X_{n+1}=j|X_n=1, D_n=k, T_{n+1}=t) = \begin{cases} 
\text{POIS}(t, i+kr-j) \quad & \text{if} \\
\max((j-i)/r, 1) \leq k \leq [(Nr-i)/r] \\
\text{POIS}(t, i+[(Nr-i)/r]r-j) \quad & \text{if} \\
\end{cases}
\]

\[
k > [(Nr-i)/r]
\]

Now we have to plug in these formulae into (13) to get the transition probabilities \( p_{ij} \). For completeness we give the results:

1) strategy is WBAS

1a) \( j > i \)

\[
p_{ij} = \sum_{k=(j-i)/r}^{[(Nr-i)/r]} \int_0^\infty \text{POIS}(t, i+kr-j) g(t) dt \alpha_k. \tag{14a}
\]

1b) \( 1 \leq j \leq i \)

\[
p_{ij} = \sum_{k=1}^{[(Nr-i)/r]} \int_0^\infty \text{POIS}(t, i-j) g(t) dt \alpha_k + \int_0^\infty \text{POIS}(t, i+[(Nr-i)/r]r-j) g(t) dt \alpha_k. \tag{14b}
\]

2) strategy is PBAS

2a) \( j > i \)

\[
p_{ij} = \sum_{k=(j-i)/r}^{[(Nr-i)/r]} \int_0^\infty \text{POIS}(t, i+kr-j) g(t) dt \alpha_k + \int_0^\infty \text{POIS}(t, i+[(Nr-i)/r]r-j) g(t) dt \alpha_k. \tag{15a}
\]

2b) \( 1 \leq j \leq i \)

\[
p_{ij} = \sum_{k=1}^{[(Nr-i)/r]} \int_0^\infty \text{POIS}(t, i+[(Nr-i)/r]r-j) g(t) dt \alpha_k. \tag{15b}
\]

At last, for \( j = 0 \) we can write

\[
p_{10} = 1 - \sum_{j=1}^{Nr} p_{ij}.
\]

Now that we have calculated the one-step transition probabilities \( p_{ij} \), we can solve the system (12) to get the desired steady-state probabilities \( z_j \). In their turn these probabilities should replace the probabilities \( f_j \) in the formulae discussed in section 1.

Next we show how the formulae for the \( p_{ij} \) can be easily evaluated for a constant interarrival-time and an Erlang-s distributed interarrival-time.
First let the interarrival-time be a constant D. Then the integrals

\[ \int_0^\infty P_0(t) s(t) g(t) dt \]

become simply \( P_0(D, n) \). Substitution of this result in the formulae (14a) to (15b) gives the transition probabilities for the deterministic case. Next we consider an Erlang-s distribution with scale parameter \( \lambda \) for the interarrival-time, i.e. the density function \( g(t) \) has the form

\[ g(t) = e^{-\lambda t} \frac{\lambda^s}{s!} t^{s-1} / (s-1)! \]

Now we can easily derive the following equality

\[ \int_0^\infty P_0(t) s(t) g(t) dt = \left( \frac{n+s-1}{s-1} \right) \times \frac{\mu^n \lambda^s}{(\mu + \lambda)^{n+s}} \]  

Substitution of (16) in the formulae (14a) to (15b) gives the \( p_{ij} \) when the interarrival-time has an Erlang-s distribution with scale parameter \( \lambda \).

In case the interarrival-time density is a mixture of Erlangian densities the integrals evaluate to a mixture of formulae of the form as in (16). So also for this type of arrival process we have simple formulae for the \( p_{ij} \).

3. The approximative approach for the \( G^X/G/1/N \) queueing model

So far we have been successful in analyzing the \( M^X/E_k/1/N \) and the \( G^X/E_k/1/N \) model. Things get worse if we allow the service-time to have a general distribution because then a Markov-chain analysis is not possible anymore. Nevertheless we can approximate performance measures such as the rejection probability and the waiting-time probabilities. This cannot be achieved directly, but should be done indirectly via percentiles. Much empirical evidence is given in [TIJMS] that the percentiles of queue size and waiting-time distribution can often be accurately approximated by interpolating the corresponding percentiles for simpler queueing systems.

It is well known that a polynomial interpolation of degree \( n \) with support points \( (x_i, f_i), i = 0, 1, \ldots, n \), is given by

\[ F(x) = \sum_{i=0}^{n} f_i \prod_{k=0}^{n} \frac{(x - x_k)}{(x_i - x_k)}. \]  

(17)
We are mostly interested in linear interpolation \((n = 1)\):  
\[ F(x) = f_0 \cdot \frac{(x-x_0)}{(x_0-x_1)} + f_1 \cdot \frac{(x-x_0)}{(x_1-x_0)}. \]

In our case we approximate the percentiles for the \(G_x/G/1/N\) queue by interpolating the percentiles for two models \(G_x/E_k/1/N\) and \(G_x/E_m/1/N\), where the interpolation is with respect to the squared coefficient of variation \(c^2\) of the service-time. Denote by \(\xi_p\) the \(p\)-th conditional waiting-time percentile for the \(G^x/G/1/N\) model i.e. \(P(W_q \leq \xi_p | W_q > 0) = p, 0<p<1\).

It is more convenient to consider the conditional waiting-time percentiles than the unconditional percentiles, since the former ones are defined for all \(0<p<1\). Then \(\xi_p\) can be approximated by
\[ \xi(k,p) * (c_k^2 - 1/m)/(1/k - 1/m) + \xi(m,p) * (c_m^2 - 1/k)/(1/m - 1/k). \]  
Here \(\xi(r,p)\) stands for the \(p\)-th conditional waiting-time percentile of the \(G^x/E_r/1/N\) model. Usually we can take \(k=1\) and \(m=2\) to get reasonable results.

An analogous approach is possible with respect to the minimal buffersize needed to assure that the rejection-probability for a customer (or a batch) does not exceed a prespecified value \(\beta\). Again we can obtain the exact values for the models \(G^x/E_k/1/N\) and \(G^x/E_m/1/N\) and use the same type of interpolation as for the waiting-time percentiles. The approximated buffersize becomes:
\[ \nu(k,\beta) * (c_k^2 - 1/m)/(1/k - 1/m) + \nu(m,\beta) * (c_m^2 - 1/k)/(1/m - 1/k). \]  
Here \(\nu(r,\beta)\) stands for the minimum buffersize for which the rejection-probability in the \(G^x/E_r/1/N\) model does not exceed a prespecified value \(\beta\).

The next question is how good these approximations are. To check this simulation is not necessary, but we can simply test the approximation for models which also allow an exact solution.

In the next section we will discuss this item.

4. Numerical results

To get a good idea of the quality of the approximation method outlined in the previous section we start with calculating the exact values of the conditional waiting-time percentiles for the following models:
\(M^x/M/1/N\), \(M^x/E_2/1/N\) and \(M^x/E_8/1/N\). In all cases we took \(E(S) = 1\).

Next we used formula (17) with \(k = 1\), \(m = 2\) and \(c_2^2 = 1/8\) to get the proposed approximative percentiles for the \(M^x/E_8/1/N\) model. In table 1a we show the results both for a constant and a geometric batch size distribution, where in either case we have taken \(E(X) = 3\). Further, the buffersize \(N = 25\) and the
rejection strategy is WBAS. We have included the results for exponential
service-times to show that the first moment is not sufficient.
Notice that the batch size distribution has a considerable effect on the
percentiles. This can also be seen from the values of the rejection
probability of an arbitrary customer. For the respective values of $\rho = 0.5,
0.8, 0.9, 1.0$ and $1.4$ these values are $0.00004$ ($0.00779$), $0.00851$ ($0.04376$),
$0.02682$ ($0.07049$), $0.06340$ ($0.10628$) and $0.28678$ ($0.29306$), where the values
between brackets correspond to the case of a geometric batch size. To get
an even better idea of the quality of the approximation we also calculated
the conditional waiting-time percentiles for the $M^X/E_2/N$ model and
we compared these with the approximations which result from formula (18)
if we use the exact results of the $M^X/M/N$ and the $M^X/E_2/N$ model.
It turned out that better approximations are obtained if we use a three-point
approximation in this case ($n = 2$ in formula (17)). We used the percentiles of
the $M^X/E_3/N$ model besides the percentiles of the earlier mentioned models.
To show the differences between the two approximations we list some results
in table 1b. Again the buffersize $N$ has been taken 25 and the mean batch size
$E(X) = 3$. We only give the results for a geometric batch size distribution and
a WBAS strategy.

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table 1b.
Comparison between the two-point and the three-point approximation for the conditional waiting-time percentiles of the $\text{M}^X/\text{E}_{20}/1/N$ model; $N = 23$, $E(X) = 3$, geometric batch size distribution

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</table>

Let us now look at the conditional waiting-time percentiles for the $\text{G}^X/\text{G}/1/N$ model. We only considered the case of constant interarrival-times and calculated the exact values for three service-time distributions: exponential service-time, Erlang-2 service-time and Erlang-8 service-time. In all cases we took $E(S)=1$. Next we used the results for the exponential case and the Erlang-2 case to get also approximative results for the $\text{D}^X/\text{E}_2/1/N$ model. We show the results in table 1c. both for a constant and a geometric batch size distribution. Also, by comparing the results with those in table 1a., notice the significant effect of the interarrival-time distribution. This can also be seen from the values for the rejection probability of an arbitrary customer.
Now we will look at the minimal buffer size for which the rejection probability of an individual customer is below a prespecified value $\beta$. Exact values can be obtained for the $M^X/E_8/1/N$ model, so we can get approximations for the minimal buffer sizes of the $M^X/G/1/N$ model via the results of the $M^X/M/1/N$ model and the $M^X/E_2/1/N$ model. So we calculated minimal buffer sizes with respect to the values $\beta = 0.01, 0.001, 0.0001$ and $0.00001$ for either model and we used the results to get approximative minimal buffer sizes for the $M^X/E_8/1/N$ model. The two point interpolations were rounded upward to the nearest integer.

In table 2a we give a comparison between the approximative and the exact values for both a geometric and a constant batch size distribution. We give only the results for the WBAS strategy.

### Table 1c

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At last we consider the minimal buffer sizes for the $G^X/G/1/N$ model. As for the waiting-time percentiles we take a constant interarrival-time and an Erlang-8 service-time distribution. Exact results have been calculated for exponential service-times and Erlang-2 service-times and these have been used to get approximations for the minimal buffer sizes of the $D^X/E_8/1/N$ model.

In table 2b, we give a comparison between these approximations and the exact results for the $D^X/E_8/1/N$ model. We give only the results for constant batch sizes and a WBAS strategy. We also include the minimal buffer sizes for the case of exponential service to show that the exact results for the case of exponential service cannot be used as first-order approximations for the case of general service. Note also from table 2b, the empirical finding that the rejection probability decreases exponentially as function of the capacity of the buffer size. It is remarked that approximations of a comparable quality were obtained when a PBAS strategy was followed. Finally we remark that for practical purposes a two-point approximation is sufficient.
table 2b

Two-point approximation for the minimal buffer sizes in the
\( \text{D}^{x}/\text{E}_8/1/N \) model; \( \mathbb{E}(X) = 3 \); strategy is WBAS

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Acknowledgement.

I would like to thank prof. dr. H. C. Tijms for his valuable guidance
during the course of this work. He suggested many improvements which made
the text much more readable.

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