Summary

This thesis consists of the text of four articles on various aspects of sandpile models, that were completed during this PhD research.

The sandpile model was introduced by Bak, Tang and Wiesenfeld to model the occurrence of power-law behavior and long-range correlations in various natural phenomena such as earthquakes, forest fires, etc. They proposed the term 'self-organized criticality' to indicate this behavior. In 'classical' criticality, well studied in statistical mechanical models, the critical state can be reached by tuning a model parameter to the critical value, such as the temperature in the Ising model. In self-organized criticality however, the idea is that the model evolves dynamically into a critical stationary state, without tuning of a model parameter. This idea is supported by extensive numerical investigations.

The motivation for the research presented in chapters 2 and 3 of this thesis is to gain a more rigorous understanding of self-organized criticality, and its relation to classical criticality. The approach has been to compare the dynamical finite volume sandpile model, better known as abelian sandpile model, to a related model which we call the infinite volume model, and which features a parameter that can be tuned. The idea behind this approach is to find a critical parameter value where a phase transition occurs in the infinite volume model, and then interpret the dynamics of the abelian sandpile model as a mechanism to steer the parameter towards this value. This would explain self-organized criticality in terms of classical criticality.

Here is a brief description of both models. The abelian sandpile starts from a stable configuration on a finite subset of $\mathbb{Z}^d$, that is, each lattice site has a number of sand grains that is strictly less than $2d$. The model evolves in discrete time. In each time step, an addition of a grain of sand is made to a random site. If the number of grains on this site is still strictly less than $2d$, it is still stable, and nothing further happens. However, if this site becomes unstable, it topples, that is, it gives a grain of sand to each neighbor (in a toppling of a boundary site, grains are lost). A toppling may cause other sites to become unstable; toppling of unstable sites continues until all sites are stable, and all topplings occur instantaneously. The model is called abelian because additions and topplings commute, so that the

order of topplings does not have to be specified.

The infinite volume model starts from a translation invariant, not necessarily stable initial configuration on $\mathbb{Z}^d$. This configuration evolves in time through topplings of unstable sites. Below, we will say more about the order of topplings. For some initial configurations, there may be a stable end configuration after only finitely many topplings per site; we call these stabilizable configurations. For others, the topplings may never result in a stable end configuration, whatever order we choose. We study stabilizability as a function of the density ρ, that is the expected amount of sand per site. The goal is to find a critical density ρ_c where a stabilizability phase transition occurs.

In Chapter 2, we prove the equivalence of two choices of toppling order, and derive bounds for the critical density. We find that $d \leq \rho_c \leq 2d-1$. In Chapter 3, we prove that, up to certain restrictions, the existence of a stable limit configuration does not depend on the order of topplings. We further prove that for $d = 1$, a configuration according to a product measure with $\rho = 1$ is not stabilizable, and moreover consider another kind of phase transition; for stabilizable configurations we study percolation of toppled sites. We prove that for sufficiently small - but nonzero - density, the clusters of toppled sites are finite.

Besides the abelian sandpile model, there exist other sandpile models that are not abelian. Chapter 4, is on Zhang's sandpile model. It differs in two aspects from the abelian sandpile model: first, an addition consists of a random, continuous amount of energy rather than one grain of sand, and second, in a toppling the entire energy content of a site is distributed in equal parts among the neighbors. Topplings and additions do not commute in general in this model. As a first result we show that in dimension 1, the topplings within each time step commute. This allows to analyze the stationary distribution of the model in dimension 1. Our main result is in the limit of infinitely many sites; here we find an explicit expression for the stationary distribution, which is such that its single site marginals concentrate on a single constant value.

Finally, in Chapter 5, we consider the abelian sandpile model as a nonrandom growth model. The initial configuration on $\mathbb{Z}^d$ consists of a constant number h of sand grains at each site. Additions are only made to the origin. The resulting configuration after a large number n of additions shows intriguing and beautiful patterns in a region around the origin. We studied this model also for negative values of h; a site with a negative number of grains can be viewed as a hole that needs to be filled up with sand.

We have studied the shape of this region in the limit $n \to \infty$. For $h = 2d-2$, we prove that this region, appropriately scaled, has a cubic limiting shape. For other fixed values of h, it is not even known whether a limiting shape exists. By comparing the sandpile model with the related rotor router model, we find that the limiting shape in the limit $h \to -\infty$ is a sphere.